Proofs in Indian Mathematics

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Abstract

Contrary to the widespread belief that Indian mathematicians did not present any proofs for their results, it is indeed the case that there is a large body of source-works in the form of commentaries which present detailed demonstrations (referred to as *upapatti*-s or *yukti*-s) for the various results enunciated in the major texts of Indian Mathematics and Astronomy. Amongst the published works, the earliest exposition of *upapatti*-s are to be found in the commentaries of Govindasvāmin (c.800) and Caturveda Pṛthūdakasvāmin (c.860). Then we find very detailed exposition of *upapatti*-s in the works of Bhāskarācārya II (c.1150). In the medieval period we have the commentaries of Śaṅkara Vāriyar (c.1535), Gaṇeśa Daivajña (c.1545), Kṛṣṇa Daivajña (c.1600) and the famous Malayalam work *Yuktibhāṣā* of Jyeṣṭhadeva (c.1530), which present detailed *upapatti*-s. By presenting a few selected examples of *upapatti*-s, we shall highlight the logical rigour which is characteristic of all the work in Indian Mathematics. We also discuss how the notion of *upapatti* is perhaps best understood in the larger epistemological perspective provided by *Nyāyaśāstra*, the Indian School of Logic. This could be of help in explicating some of the important differences between the notion of *upapatti* and the notion of "proof" developed in the Greco-European tradition of Mathematics.

1 Alleged Absence of Proofs in Indian Mathematics

Several books have been written on the history of Indian tradition in mathematics.[1] In addition, many books on history of mathematics devote a section, sometimes even a chapter, to the discussion of Indian mathematics. Many of the results and algorithms discovered by the Indian mathematicians have been studied in some detail. But, little attention has been paid to the methodology and foundations of Indian mathematics. There is hardly any discussion of the processes by which Indian mathematicians arrive at and justify their results and procedures. And, almost no attention is paid to the philosophical foundations of Indian mathematics, and the Indian understanding of the nature of mathematical objects, and validation of mathematical results and procedures.

Many of the scholarly works on history of mathematics assert that Indian Mathematics, whatever its achievements, does not have any sense of logical rigour. Indeed, a major historian of mathematics presented the following assessment of Indian mathematics over fifty years ago:

> The Hindus apparently were attracted by the arithmetical and computational aspects of mathematics rather than by the geometrical and rational features of the subject which had appealed so strongly to the Hellenistic mind. Their name for mathematics, *gaṇita*, meaning literally the 'science of calculation' well characterizes this preference. They delighted more in the tricks that could be played with numbers than in the thoughts the mind could produce, so that neither Euclidean geometry nor Aristotelian logic made a strong impression upon them. The Pythagorean problem of the incommensurables, which was of intense interest to Greek geometers, was of little import to Hindu mathematicians, who treated rational and irrational quantities, curvilinear

[1] We may cite the following standard works: B.B.Datta and A.N.Singh, *History of Hindu Mathematics,* 2 parts, Lahore 1935, 1938, Reprint, Delhi 1962; C.N.Srinivasa Iyengar, *History of Indian Mathematics,* Calcutta 1967; A.K.Bag, *Mathematics in Ancient and Medieval India,* Varanasi 1979; T.A.Saraswati Amma, *Geometry in Ancient and Medieval India*, Varanasi 1979; G.C.Joseph, *The Crest of the Peacock: The Non-European Roots of Mathematics,* 2^{nd} Ed., Princeton 2000.

> and rectilinear magnitudes indiscriminately. With respect to the development of algebra, this attitude occasioned perhaps an incremental advance, since by the Hindus the irrational roots of the quadratics were no longer disregarded as they had been by the Greeks, and since to the Hindus we owe also the immensely convenient concept of the absolute negative. These generalizations of the number system and the consequent freedom of arithmetic from geometrical representation were to be essential in the development of the concepts of calculus, but the Hindus could hardly have appreciated the theoretical significance of the change...
>
> The strong Greek distinction between the discreteness of number and the continuity of geometrical magnitude was not recognized, for it was superfluous to men who were not bothered by the paradoxes of Zeno or his dialectic. Questions concerning incommensurability, the infinitesimal, infinity, the process of exhaustion, and the other inquiries leading toward the conceptions and methods of calculus were neglected.[2]

Such views have found their way generally into more popular works on history of mathematics. For instance, we may cite the following as being typical of the kind of opinions commonly expressed about Indian mathematics:

> As our survey indicates, the Hindus were interested in and contributed to the arithmetical and computational activities of mathematics rather than to the deductive patterns. Their

[2]C.B.Boyer, *The History of Calculus and its Conceptual development,* New York 1949, p.61-62. As we shall see in the course of this article, Boyer's assessment – that the Indian mathematicians did not reach anywhere near the development of calculus or mathematical analysis, because they lacked the sophisticated methodology developed by the Greeks – seems to be thoroughly misconceived. In fact, in marked contrast to the development of mathematics in the Greco-European tradition, the methodology of Indian mathematical tradition seems to have ensured continued and significant progress in all branches of mathematics till barely two hundred year ago; it also lead to major discoveries in calculus or mathematical analysis, without in anyway abandoning or even diluting its standards of logical rigour, so that these results, and the methods by which they were obtained, seem as much valid today as at the time of their discovery.

> name for mathematics was *gaṇita*, which means "the science of calculation". There is much good procedure and technical facility, but no evidence that they considered proof at all. They had rules, but apparently no logical scruples. Moreover, no general methods or new viewpoints were arrived at in any area of mathematics.
>
> It is fairly certain that the Hindus did not appreciate the significance of their own contributions. The few good ideas they had, such as separate symbols for the numbers from 1 to 9, the conversion to base 10, and negative numbers, were introduced casually with no realization that they were valuable innovations. They were not sensitive to mathematical values. Along with the ideas they themselves advanced, they accepted and incorporated the crudest ideas of the Egyptians and Babylonians.[3]

The burden of scholarly opinion is such that even eminent mathematicians, many of whom have had fairly close interaction with contemporary Indian mathematics, have ended up subscribing to similar views, as may be seen from the following remarks of one of the towering figures of twentieth century mathematics:

> For the Indians, of course, the effectiveness of the *cakravāla* could be no more than an experimental fact, based on their treatment of great many specific cases, some of them of considerable complexity and involving (to their delight, no doubt) quite large numbers. As we shall see, Fermat was the first one to perceive the need for a general proof, and Lagrange was the first to publish one. Nevertheless, to have developed the *cakravāla* and to have applied it successfully to such difficult numerical cases as $N = 61$, or $N = 67$ had been no mean achievements.[4]

[3]Morris Kline, *Mathematical Thought from Ancient to Modern Times,* Oxford 1972, p.190.

[4]Andre Weil, *Number Theory: An Approach through History from Hammurapi to Legendre,* Boston 1984, p.24. It is indeed ironical that Prof. Weil has credited Fermat, who is notorious for not presenting proofs for most of the claims he made, with the realization that mathematical results need to be justified by proofs. While the rest of this article is purported to show that the Indian mathematicians presented logically

Modern scholarship seems to be unanimous in holding the view that Indian mathematics is bereft of any notion of proof. But even a cursory study of the source-works that are available in print would reveal that Indian mathematicians place much emphasis on providing what they refer to as *upapatti* (proof, demonstration) for every one of their results and procedures. Some of these *upapatti*-s were noted in the early European studies on Indian mathematics in the first half of the nineteenth century. For instance, in 1817, H.T. Colebrooke notes the following in the preface to his widely circulated translation of portions of *Brāhmasphuṭasiddhānta* of Brahmagupta and *Līlāvatī* and *Bījagaṇita* of Bhāskarācārya:

> On the subject of demonstrations, it is to be remarked that the Hindu mathematicians proved propositions both algebraically and geometrically: as is particularly noticed by Bhāskara himself, towards the close of his algebra, where he gives both modes of proof of a remarkable method for the solution of indeterminate problems, which involve a factum of two unknown quantities.[5]

Another notice of the fact that detailed proofs are provided in the Indian texts on mathematics is due to C.M.Whish who, in an article published in 1835, pointed out that infinite series for π and for trigonometric functions were derived in texts of Indian mathematics much before their 'discovery' in Europe. Whish concluded his paper with a sample proof

rigorous proofs for most of the results and processes that they discovered, it must be admitted that the particular example that Prof. Weil is referring to, the effectiveness of the *cakravāla* algorithm (known to the Indian mathematicians at least from the time of Jayadeva, prior to the eleventh century) for solving quadratic indeterminate equations of the form $x^2 - Ny^2 = 1$, does not seem to have been demonstrated in the available source-works. In fact, the first proof of this result was given by Krishnaswamy Ayyangar barely seventy-five years ago (A.A.Krishnaswamy Ayyangar, "New Light on Bhāskara's *Cakravāla* or Cyclic Method of solving Indeterminate Equations of the Second Degree in Two Variables', Jour. Ind. Math. Soc. **18**, 228-248, 1929-30). Krishnaswamy Ayyangar also showed that the *cakravāla* algorithm is different and more optimal than the Brouncker-Wallis-Euler-Lagrange algorithm for solving this so-called "Pell's Equation."

[5]H.T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhāskara,* London 1817, p.xvii. Colebrooke also presents some of the *upapatti*-s given by the commentators Gaṇeśa Daivajña and Kṛṣṇa Daivajña, as footnotes in his work.

from the Malayalam text *Yuktibhāṣā* of the theorem on the square of the hypotenuse of a right angled triangle and also promised that:

> A further account of the *Yuktibhāṣā*, the demonstrations of the rules for the quadrature of the circle of infinite series, with the series for the sines, cosines, and their demonstrations, will be given in a separate paper: I shall therefore conclude this, by submitting a simple and curious proof of the 47^{th} proposition of Euclid [the so called Pythagoras theorem], extracted from the *Yuktibhāṣā*.[6]

It would indeed be interesting to find out how the currently prevalent view, that Indian mathematics lacks the notion of proof, obtained currency in the last 100-150 years.

2 *Upapatti*-s in Indian Mathematics

2.1 The tradition of *Upapatti*-s in Mathematics and Astronomy

A major reason for our lack of comprehension, not merely of the Indian notion of proof, but also of the entire methodology of Indian mathematics, is the scant attention paid to the source-works so far. It is said that there are over one hundred thousand manuscripts on *Jyotiḥśāstra*, which includes, apart from works on *gaṇita* (mathematics and mathematical astronomy), also those on *saṃhitā* (omens) and *hora* (astrology).[7] Only a small fraction of these texts have been published. A well known source book, lists about 285 published works in mathematics and mathematical astronomy. Of these, about 50 are from the period before 12^{th} century AD, about 75 from $12^{th} - 15^{th}$ centuries, and about 165 from $16^{th} - 19^{th}$

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[6] C.M. Whish, 'On the Hindu Quadrature of the Circle, and the Infinite Series of the Proportion of the Circumference to the Diameter Exhibited in the Four Shastras, the Tantrasangraham, Yucti Bhasa, Carana Paddhati and Sadratnamala', Trans.Roy.As.Soc.(G.B.) **3**, 509-523, 1835. However, Whish does not seem to have published any further paper on this subject.

[7] D. Pingree, *Jyotiḥśāstra*: *Astral and Mathematical Literature*, Wiesbaden 1981, p.118.

centuries.[8]

Much of the methodological discussion is usually contained in the detailed commentaries; the original works rarely touch upon such issues. Modern scholarship has concentrated on translating and analysing the original works alone, without paying much heed to the commentaries. Traditionally, the commentaries have played at least as great a role in the exposition of the subject as the original texts. Great mathematicians and astronomers, of the stature of Bhāskarācārya I, Bhāskarācārya II, Parameśvara, Nīlakaṇṭha Somasutvan, Gaṇeśa Daivajña, Munīśvara and Kamalākara, who wrote major original treatises of their own, also took great pains to write erudite commentaries on their own works and on the works of earlier scholars. It is in these commentaries that one finds detailed *upapatti*-s of the results and procedures discussed in the original texts, as also a discussion of the various methodological and philosophical issues. For instance, at the beginning of his commentary *Buddhivilāsinī*, Gaṇeśa Daivajña states:

> श्रीभास्करोक्तवचसामपि संस्फुटानां
> व्याख्याविशेषकथनेन न चास्ति चित्रम् ।
> अत्रोपपत्तिकथनेऽखिलसारभूते
> पश्यन्तु सुज्ञगणका मम बुद्धिचित्रम् ॥

> There is no purpose served in providing further explanations for the already lucid statements of Śrī Bhāskara. The knowledgeable mathematicians may therefore note the specialty of my intellect in the statement of *upapatti*-s, which are after all the essence of the whole thing.[9]

Amongst the published works on Indian mathematics and astronomy, the earliest exposition of *upapatti*-s are to be found in the *bhāṣya* of Govindasvāmin (c 800) on *Mahābhāskarīya* of Bhāskarācārya I, and the *Vāsanābhāṣya* of Caturveda Pṛthūdakasvāmin (c 860) on *Brāhma-*

[8] K.V. Sarma and B.V. Subbarayappa, *Indian Astronomy: A Source Book,* Bombay 1985.

[9] *Buddhivilāsinī* of Gaṇeśa Daivajña, V.G. Apte (ed.), Vol I, Pune 1937, p.3.

sphuṭasiddhānta of Brahmagupta.[10] Then we find very detailed exposition of *upapatti*-s in the works of Bhāskarācārya II (c.1150): his *Vivaraṇa* on *Śiṣyadhīvṛddhidatantra* of Lalla and *Vāsanābhāṣya* on his own *Siddhāntaśiromaṇi*.[11] Apart from these, Bhāskarācārya provides an idea of what is an *upapatti* in his *Bījavāsanā* on his own *Bījagaṇita* in two places. In the chapter on *madhyamāharaṇa* (quadratic equations) he poses the following problem:

> Find the hypotenuse of a plane figure, in which the side and upright are equal to fifteen and twenty. And show the *upapatti* (demonstration) of the standard procedure of computation.[12]

Bhāskarācārya provides two *upapatti*-s for the solution of this problem, the so-called Pythagoras theorem; and we shall consider them later. Again, towards the end of the *Bījagaṇita* in the chapter on *bhāvita* (equations involving products), while considering integral solutions of equations of the form $ax + by = cxy$, Bhāskarācārya explains the nature of *upapatti* with the help of an example:

> The *upapatti* (demonstration) follows. It is twofold in each case: One geometric and the other algebraic. The geometric demonstration is here presented... The algebraic demonstration is next set forth... This procedure [of demonstration] has been earlier presented in a concise instructional form [*saṃkṣiptapāṭha*] by ancient teachers. The algebraic demonstrations are for those who do not comprehend the geometric one. Mathematicians have said that algebra is computation joined with demonstration; otherwise there would be no difference between arithmetic and algebra. Therefore this demonstration of *bhāvita* has been shown in two ways.[13]

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[10]The *Āryabhaṭīya-bhāṣya* of Bhāskara I (c.629) does occasionally indicate the derivation of some of the mathematical procedures, though his commentary does not purport to present *upapatti*-s for the rules and procedures given in *Āryabhaṭīya*.

[11]This latter commentary of Bhāskara II is a classic source of *upapatti*-s and needs to be studied in depth.

[12]*Bījagaṇita* of Bhāskarācārya, Muralidhara Jha (ed.), Varanasi 1927, p.69.

[13]*Bījagaṇita*, cited above, p.125-127.

Clearly the tradition of exposition of *upapatti*-s is much older and Bhāskarācārya, and the later mathematicians and astronomers are merely following the traditional practice of providing detailed *upapatti*-s in their commentaries to earlier, or their own, works.[14]

In Appendix A we give a list of important commentaries, available in print, which present detailed *upapatti*-s. It is unfortunate that none of the published source-works that we have mentioned above has so far been translated into any of the Indian languages, or into English; nor have they been studied in depth with a view to analyze the nature of mathematical arguments employed in the *upapatti*-s or to comprehend the methodological and philosophical foundations of Indian mathematics and astronomy.[15]

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[14]Ignoring all these classical works on *upapatti*-s, one scholar has recently claimed that the tradition of *upapatti* in India "dates from the 16^{th} and 17^{th} centuries" (J.Bronkhorst, '*Pāṇini* and Euclid', Jour. Ind. Phil. **29**, 43-80, 2001).

[15]We may, however, mention the following works of C.T.Rajagopal and his collaborators which discuss some of the *upapatti*-s presented in the Malayalam work *Yuktibhāṣā* of Jyeṣṭhadeva (c.1530) for various results in geometry, trigonometry and those concerning infinite series for π and the trigonometric functions: K. Mukunda Marar, 'Proof of Gregory's Series', Teacher's Magazine **15**, 28-34, 1940; K. Mukunda Marar and C.T.Rajagopal, 'On the Hindu Quadrature of the Circle', J.B.B.R.A.S. **20**, 65-82, 1944; K. Mukunda Marar and C.T.Rajagopal, 'Gregory's Series in the Mathematical Literature of Kerala', Math Student **13**, 92-98, 1945; A. Venkataraman, 'Some Interesting Proofs from *Yuktibhāṣā* ', Math Student **16**, 1-7, 1948; C.T.Rajagopal 'A Neglected Chapter of Hindu Mathematics', Scr. Math. **15**, 201-209, 1949; C.T.Rajgopal and A. Venkataraman, 'The Sine and Cosine Power Series in Hindu Mathematics', J.R.A.S.B. **15**, 1-13, 1949; C.T. Rajagopal and T.V.V.Aiyar, 'On the Hindu Proof of Gregory's Series', Scr. Math. **17**, 65-74, 1951; C.T.Rajagopal and T.V.V.Aiyar, 'A Hindu Approximation to Pi', Scr.Math. **18**, 25-30, 1952. C.T.Rajagopal and M.S.Rangachari, 'On an Untapped Source of Medieval Keralese Mathematics', Arch. for Hist. of Ex. Sc. **18**, 89-101, 1978; C.T.Rajagopal and M.S.Rangachari, 'On Medieval Kerala Mathematics', Arch. for Hist. of Ex. Sc. **35**(2), 91-99, 1986.

Following the work of Rajagopal and his collaborators, there are some recent studies which discuss some of the proofs in *Yuktibhāṣā* . We may here cite the following: T.Hayashi, T.Kusuba and M.Yano, 'The Correction of the Mādhava Series for the Circumference of a Circle', Centauras, **33**, 149-174, 1990; Ranjan Roy, 'The Discovery of the Series formula for π by Leibniz, Gregory and Nīlakaṇṭha', Math. Mag. **63**, 291-306, 1990; V.J.Katz, 'Ideas of Calculus in Islam and India', Math. Mag. **68**, 163-174, 1995; C.K.Raju, 'Computers, Mathematics Education, and the Alternative Epistemology of the Calculus in the *Yuktibhāṣā* ', Phil. East and West **51**, 325-362, 2001; D.F.Almeida, J.K.John and A.Zadorozhnyy, 'Keralese Mathematics: Its Possible Transmission to Europe and the Consequential Educational Implications', In this article, we shall present some examples of the kinds of *upapatti*-s provided in Indian mathematics, from the commentaries of Gaṇeśa Daivajña (c.1545) and Kṛṣṇa Daivajña (c.1600) on the texts *Līlāvatī* and *Bījagaṇita* respectively, of Bhāskarācārya II (c.1150), and from the celebrated Malayalam work *Yuktibhāṣā* of Jyeṣṭhadeva (c.1530). We shall also discuss how the notion of *upapatti* is perhaps best understood in the larger epistemological perspective provided by *Nyāya-śāstra* the Indian School of Logic. This enables us to explicate some of the important differences between the notion of *upapatti* and the notion of "proof" developed in the Greco-European tradition of Mathematics.

2.2 Mathematical results should be supported by *Upapatti*-s

Before discussing some of the *upapatti*-s presented in Indian mathematical tradition, it is perhaps necessary to put to rest the widely prevalent myth that the Indian mathematicians did not pay any attention to, and perhaps did not even recognize the need for justifying the mathematical results and procedures that they employed. The large corpus of *upapatti*-s, even amongst the small sample of source-works published so far, should convince anyone that there is no substance to this myth. Still, we may cite the following passage from Kṛṣṇa Daivajña's commentary *Bījapallava* on *Bījagaṇita* of Bhāskarācārya, which clearly brings out the basic understanding of Indian mathematical tradition that citing any number of instances (even an infinite number of them) where a particular result seems to hold, does not amount to establishing that as a valid result in mathematics; only when the result is supported by a *upapatti* or a demonstration, can the result be accepted as valid:

> How can we state without proof (*upapatti*) that twice the product of two quantities when added or subtracted from

J. Nat. Geo. **20**, 77-104, 2001; D.Bressoud, 'Was Calculus Invented in India?', College Math. J. **33**, 2-13, 2002; J.K.John, 'Derivation of the *Saṃskāras* applied to the Mādhava Series in *Yuktibhāṣā* ', in M.S.Sriram, K.Ramasubramanian and M.D.Srinivas (eds.), *500 Years of Tantrasaṅgraha : A Landmark in the History of Astronomy*, Shimla 2002, p 169-182. An outline of the proofs given in *Yuktibhāṣā* can also be found in T.A. Saraswati Amma, 1979, cited earlier, and in S.Parameswaran, *The Golden Age of Indian Mathematics,* Kochi 1998.

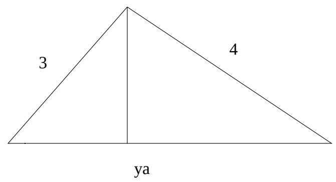
the sum of their squares is equal to the square of the sum or difference of those quantities? That it is seen to be so in a few instances is indeed of no consequence. Otherwise, even the statement that four times the product of two quantities is equal to the square of their sum, would have to be accepted as valid. For, that is also seen to be true in some cases. For instance, take the numbers 2, 2. Their product is 4, four times which will be 16, which is also the square of their sum 4. Or take the numbers 3, 3. Four times their product is 36, which is also the square of their sum 6. Or take the numbers 4, 4. Their product is 16, which when multiplied by four gives 64, which is also the square of their sum 8. Hence, the fact that a result is seen to be true in some cases is of no consequence, as it is possible that one would come across contrary instances (*vyabhicāra*) also. Hence it is necessary that one would have to provide a proof (*yukti*) for the rule that twice the product of two quantities when added or subtracted from the sum of their squares results in the square of the sum or difference of those quantities. We shall provide the proof (*upapatti*) in the end of the section on *ekavarṇa-madhyamāharaṇa*.[16]

2.3 Square of the hypotenuse of a right-angled triangle: the so-called Pythagoras Theorem

Gaṇeśa provides two *upapatti*-s for the rule concerning the square of the hypotenuse (*karṇa*) of a right-angled triangle.[17] These *upapatti*-s are the same as the ones outlined by Bhāskarācārya II in his *Bījavāsanā* on his own *Bījagaṇita*, that we referred to earlier. The first involves the *avyakta* method and proceeds as follows:

[16] *Bījapallava* of Kṛṣṇa Daivajña, T.V. Radhakrishna Sastri (ed.), Tanjore, 1958, p.54.

[17] *Buddhivilāsinī*, cited earlier, p.128-129.



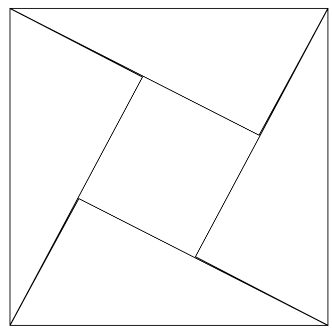
$$
\begin{aligned}
y\bar{a} &= \left(\frac{9}{y\bar{a}}\right) + \left(\frac{16}{y\bar{a}}\right) \\
y\bar{a}^2 &= 25 \\
y\bar{a} &= 5
\end{aligned}
$$

Take the hypotenuse (*karṇa*) as the base and assume it to be *yā*. Let the *bhujā* and *koṭi* (the two sides) be 3 and 4 respectively. Take the hypotenuse as the base and draw the perpendicular to the hypotenuse from the opposite vertex as in the figure. [This divides the triangle into two triangles, which are similar to the original] Now by the rule of proportion (*anupāta*), if *yā* is the hypotenuse the *bhujā* is 3, then when this *bhujā* 3 is the hypotenuse, the *bhujā*, which is now the *ābādhā* (segment of the base) on the side of the original *bhujā* will be $(\frac{9}{y\bar{a}})$.

Again if *yā* is the hypotenuse, the *koṭi* is 4, then when this *koṭi* 4 is the hypotenuse, the *koṭi*, which is now the segment of base on the side of the (original) *koṭi* will be $(\frac{16}{y\bar{a}})$. Adding the two segments (*ābādhā*-s) of *yā* the hypotenuse and equating the sum to (the hypotenuse) *yā*, cross-multiplying and taking the square-roots, we get $y\bar{a} = 5$, the square root of the sum of the squares of *bhujā* and *koṭi* .

The other *upapatti* of Gaṇeśa is *kṣetragata* or geometrical, and proceeds as follows:[18]

[18]This method seems to be known to Bhāskarācārya I (c.629 AD) who gives a very similar diagram in his *Āryabhaṭīyabhāṣya*, K.S. Shukla (ed.), Delhi 1976, p.48. The Chinese mathematician Liu Hui (c 3^{rd} century AD) seems to have proposed similar geometrical proofs of this so-called Pythagoras Theorem. See for instance, D.B.Wagner, ‘A Proof of the Pythagorean Theorem by Liu Hui’, Hist. Math.**12**, 71-3, 1985.



$$c^2 = (a-b)^2 + 4(\tfrac{1}{2}ab)$$
$$= a^2 + b^2$$

Take four triangles identical to the given and taking the four hypotenuses to be the four sides, form the square as shown. Now, the interior square has for its side the difference of *bhujā* and *koṭi* . The area of each triangle is half the product of *bhujā* and *koṭi* and four times this added to the area of the interior square is the area of the total figure. This is twice the product of *bhujā* and *koṭi* added to the square of their difference. This, by the earlier cited rule, is nothing but the sum of the squares of *bhujā* and *koṭi* . The square root of that is the side of the (big) square, which is nothing but the hypotenuse.

2.4 The rule of signs in Algebra

One of the important aspects of Indian mathematics is that in many *upapatti*-s the nature of the underlying mathematical objects plays an important role. We can for instance, refer to the *upapatti* given by Kṛṣṇa Daivajña for the well-known rule of signs in Algebra. While providing an *upapatti* for the rule, "the number to be subtracted if positive (*dhana*) is made negative (*ṛṇa*) and if negative is made positive", Kṛṣṇa Daivajña states:

> Negativity (*ṛṇatva*) here is of three types – spatial, temporal and that pertaining to objects. In each case, it [negativity] is indeed the *vaiparītya* or the oppositeness...For instance, the other direction in a line is called the opposite direction (*viparīta dik*); just as west is the opposite of east... Further, between two stations if one way of traversing is considered positive then the other is negative. In the same way, past and future time intervals will be mutually negative

> of each other... Similarly, when one possesses said objects they would be called his *dhana* (wealth). The opposite would be the case when another owns the same objects... Amongst these [different conceptions], we proceed to state the *upapatti* of the above rule, assuming positivity (*dhanatva*) for locations in the eastern direction and negativity (*ṛṇatva*) for locations in the west, as follows...[19]

Kṛṣṇa Daivajña goes on to explain how the distance between a pair of stations can be computed knowing that between each of these stations and some other station on the same line. Using this he demonstrates the above rule that "the number to be subtracted if positive is made negative..."

2.5 The *Kuṭṭaka* process for the solution of linear indeterminate equations

To understand the nature of *upapatti* in Indian mathematics one will have to analyse some of the lengthy demonstrations which are presented for the more complicated results and procedures. One will also have to analyse the sequence in which the results and the demonstrations are arranged to understand the method of exposition and logical sequence of arguments. For instance, we may refer to the demonstration given by Kṛṣṇa Daivajña[20] of the well-known *kuṭṭaka* procedure, which has been employed by Indian mathematicians at least since the time of Āryabhaṭa (c 499 AD), for solving first order indeterminate equations of the form

$$\frac{(ax+c)}{b} = y,$$

where a, b, c are given integers and x, y are to be solved for in integers. Since this *upapatti* is rather lengthy, we merely recount the essential steps here.[21]

[19] *Bījapallava*, cited above, p.13.

[20] *Bījapallava*, cited above, p.85-99.

[21] A translation of the *upapatti* may be found in M.D.Srinivas, 'Methodology of Indian Mathematics and its Contemporary Relevance', PPST Bulletin, **12**, 1-35, 1987.

Kṛṣṇa Daivajña first shows that the solutions for x, y do not vary if we factor all the three numbers $a,\ b,\ c$ by the same common factor. He then shows that if a and b have a common factor, then the above equation will not have a solution unless c is also divisible by the same common factor. Then follows the *upapatti* of the process of finding the greatest common factor of a and b by mutual division, the so-called Euclidean algorithm. He then provides an *upapatti* for the *kuṭṭaka* method of finding the solution which involves carrying out a sequence of transformations on the *vallī* (line or column) of quotients obtained in the above mutual division. This is based on a detailed analysis of the various operations in reverse (*vyasta-vidhi*). The last two elements of the *vallī*, at each stage, are shown to be the solutions of the *kuṭṭaka* problem involving the successive pair of remainders (taken in reverse order from the end) which arise in the mutual division of a and b. Finally, it is shown how the procedure differs depending upon whether there are odd or even number of coefficients generated in the above mutual division.

2.6 Nīlakaṇṭha's proof for the sum of an infinite geometric series

In his *Āryabhaṭīyabhāṣya* while deriving an interesting approximation for the arc of circle in terms of the *jyā* (Rsine) and the *śara* (Rversine), the celebrated Kerala astronomer Nīlakaṇṭha Somasutvan presents a detailed demonstration of how to sum an infinite geometric series. Though it is quite elementary compared to the various other infinite series expansions derived in the works of the Kerala School, we shall present an outline of Nīlakaṇṭha's argument as it clearly shows how the notion of limit was well understood in the Indian mathematical tradition. Nīlakaṇṭha first states the general result[22]

$$a\left[\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)^2+\left(\frac{1}{r}\right)^3+\ldots\right]=\frac{a}{r-1}\,.$$

where the left hand side is an infinite geometric series with the successive terms being obtained by dividing by a *cheda* (common divisor), r, assumed to be greater than 1. Nīlakaṇṭha notes that this result is best

[22] *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha, *Gaṇitapāda*, K.Sambasiva Sastri (ed.), Trivandrum 1931, p.142-143.

demonstrated by considering a particular case, say $r = 4$. Thus, what is to be demonstrated is that

$$\left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \ldots = \frac{1}{3}\,.$$

Nīlakaṇṭha first obtains the sequence of results

$$\begin{aligned}
\frac{1}{3} &= \frac{1}{4} + \frac{1}{(4.3)}\,, \\
\frac{1}{(4.3)} &= \frac{1}{(4.4)} + \frac{1}{(4.4.3)}\,, \\
\frac{1}{(4.4.3)} &= \frac{1}{(4.4.4)} + \frac{1}{(4.4.4.3)}\,,
\end{aligned}$$

and so on, from which he derives the general result

$$\frac{1}{3} - \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n\right] = \left(\frac{1}{4}\right)^n \left(\frac{1}{3}\right).$$

Nīlakaṇṭha then goes on to present the following crucial argument to derive the sum of the infinite geometric series: As we sum more terms, the difference between $\frac{1}{3}$ and sum of powers of $\frac{1}{4}$ (as given by the right hand side of the above equation), becomes extremely small, but never zero. Only when we take all the terms of the infinite series together do we obtain the equality

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \ldots + \left(\frac{1}{4}\right)^n + \ldots = \frac{1}{3}\,.$$

2.7 *Yuktibhāṣā* proofs of infinite series for π and the trigonometric functions

One of the most celebrated works in Indian mathematics and astronomy, which is especially devoted to the exposition of *yukti* or proofs, is the Malayalam work *Yuktibhāṣā* (c.1530) of Jyeṣṭhadeva.[23] Jyeṣṭhadeva

[23] *Yuktibhāṣā* of Jyeṣṭhadeva, K. Chandrasekharan (ed.), Madras 1953. *Gaṇitādhyāya* alone was edited along with notes in Malayalam by Ramavarma Thampuran and A.R.Akhileswara Aiyer, Trichur 1948. The entire work has been edited, along with an ancient Sanskrit version, *Gaṇitayuktibhāṣā* and English translation, by K.V.Sarma, with explanatory mathematical notes by K.Ramasubramanian, M.D.Srinivas and M.S.Sriram (in press).

states that his work closely follows the renowned astronomical work *Tantrasaṅgraha* (c.1500) of Nīlakaṇṭha Somasutvan and is intended to give a detailed exposition of all the mathematics required thereof. The first half of *Yuktibhāṣā* deals with various mathematical topics in seven chapters and the second half deals with all aspects of mathematical astronomy in eight chapters. The mathematical part includes a detailed exposition of proofs for the infinite series and fast converging approximations for π and the trigonometric functions, which were discovered by Mādhava (c.1375). We present an outline of some of these proofs in Appendix B.

3 *Upapatti* and "Proof"

3.1 Mathematics as a search for infallible eternal truths

The notion of *upapatti* is significantly different from the notion of 'proof' as understood in the Greek as well as the modern Western tradition of mathematics. The ideal of mathematics in the Greek and modern Western traditions is that of a formal axiomatic deductive system; it is believed that mathematics is and ought to be presented as a set of formal derivations from formally stated axioms. This ideal of mathematics is intimately linked with another philosophical presupposition – that mathematics constitutes a body of infallible eternal truths. Perhaps it is only the ideal of a formal axiomatic deductive system that could presumably measure up to this other ideal of mathematics being a body of infallible eternal truths. It is this quest for securing certainty of mathematical knowledge, which has motivated most of the foundational and philosophical investigations into mathematics and shaped the course of mathematics in the Western tradition, from the Greeks to the contemporary times.

The Greek view of mathematical objects and the nature of mathematical knowledge is clearly set forth in the following statement of Proclus (c. 5^{th} century AD) in his famous commentary on the Elements of Euclid:

> Mathematical being necessarily belongs neither among the first nor among the last and least simple kinds of being, but occupies the middle ground between the partless realities –

> simple, incomposite and indivisible – and divisible things characterized by every variety of composition and differentiation. The unchangeable, stable and incontrovertible character of the propositions about it shows that it is superior to the kind of things that move about in matter ...
>
> It is for this reason, I think, that Plato assigned different types of knowing to the highest, the intermediate, and the lowest grades of reality. To indivisible realities he assigned intellect, which discerns what is intelligible with simplicity and immediacy, and by its freedom from matter, its purity, and its uniform mode of coming in contact with being is superior to all other forms of knowledge. To divisible things in the lowest level of nature, that is, to all objects of sense perception, he assigned opinion, which lays hold of truth obscurely, whereas to intermediates, such as the forms studied by mathematics, which fall short of indivisible but are superior to divisible nature, he assigned understanding....
>
> Hence Socrates describes the knowledge of understandables as being more obscure than the highest science but clearer than the judgements of opinion. For, the mathematical sciences are more explicative and discursive than intellectual insight but are superior to opinion in the stability and irrefutability of their ideas. And their proceeding from hypothesis makes them inferior to highest knowledge, while their occupation with immaterial objects makes their knowledge more perfect than sense perception.[24]

While the above statement of Proclus is from the Platonist school, the Aristotelean tradition also held more or less similar views on the nature of mathematical knowledge, as may be seen from the following extract from the canonical text on Mathematical Astronomy, the Almagest of Claudius Ptolemy (c.2^{nd} century AD):

> For Aristotle divides theoretical philosophy too,very fittingly, into three primary categories, physics, mathematics and theology. For everything that exists is composed of matter, form

[24]Proclus: *A Commentary on the First Book of Euclid's Elements*, Tr.G.R.Morrow, Princeton, 1970, p.3,10.

and motion; none of these [three] can be observed in its substratum by itself, without the others: they can only be imagined. Now the first cause of the first motion of the universe, if one considers it simply, can be thought of as invisible and motionless deity; the division [of theoretical philosophy] concerned with investigating this [can be called] 'theology', since this kind of activity, somewhere up in the highest reaches of the universe, can only be imagined, and is completely separated from perceptible reality. The division [of theoretical philosophy] which investigates material and ever-moving nature, and which concerns itself with 'white', 'hot', 'sweet', 'soft' and suchlike qualities one may call 'physics'; such an order of being is situated (for the most part) amongst corruptible bodies and below the lunar sphere. That division [of theoretical philosophy] which determines the nature involved in forms and motion from place to place, and which serves to investigate shape, number, size and place, time and suchlike, one may define as 'mathematics'. Its subject-matter falls as it were in the middle between the other two, since, firstly, it can be conceived of both with and without the aid of the senses, and, secondly, it is an attribute of all existing things without exception, both mortal and immortal: for those things which are perpetually changing in their inseparable form, it changes with them, while for eternal things which have an aethereal nature, it keeps their unchanging form unchanged.

From all this we concluded: that the first two divisions of theoretical philosophy should rather be called guesswork than knowledge, theology because of its completely invisible and ungraspable nature, physics because of the unstable and unclear nature of matter; hence there is no hope that philosophers will ever be agreed about them; and that only mathematics can provide sure and unshakeable knowledge to its devotees, provided one approaches it rigorously. For its kind of proof proceeds by indisputable methods, namely arithmetic and geometry. Hence we are drawn to the investigation of that part of theoretical philosophy, as far as we were able to the whole of it, but especially to the theory concerning

> the divine and heavenly things. For that alone is devoted to the investigation of the eternally unchanging. For that reason it too can be eternal and unchanging (which is a proper attribute of knowledge) in its own domain, which is neither unclear nor disorderly.[25]

The view, that it is mathematics which can provide "sure and unshakeable knowledge to its devotees" has persisted in the Greco-European tradition down to the modern times. For instance, we may cite the popular mathematician philosopher of our times, Bertrand Russel, who declares, "I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere". In a similar vein, David Hilbert, one of the foremost mathematicians of our times declared, "The goal of my theory is to establish once and for all the certitude of mathematical methods".[26]

3.2 The *raison d'être* of *Upapatti*

Indian epistemological position on the nature and validation of mathematical knowledge is very different from that in the Western tradition. This is brought out for instance by the Indian understanding of what indeed is the purpose or *raison d'être* of an *upapatti*. In the beginning of the *golādhyāya* of *Siddhāntaśiromaṇi*, Bhāskarācārya says:

> मध्याद्यं द्युसदां यदत्र गणितं तस्योपपत्तिं विना
> प्रौढिं प्रौढसभासु नैति गणको निःसंशयो न स्वयम् ।
> गोले सा विमला करामलकवत् प्रत्यक्षतो दृश्यते
> तस्मादस्म्युपपत्तिबोधविधये गोलप्रबन्धोद्यतः ॥[27]

> Without the knowledge of *upapatti*-s, by merely mastering the *gaṇita* (calculational procedures) described here, from

[25] *The Almagest* of Ptolemy, Translated by G.J.Toomer, London 1984, p.36-7.

[26] Both quotations cited in Ruben Hersh, 'Some Proposals for Reviving the Philosophy of Mathematics', Adv. Math. **31**, 31-50, 1979.

[27] *Siddhāntaśiromaṇi* of Bhāskarācārya with *Vāsanābhāṣya* and *Vāsanāvārttika* of Nṛsiṃha Daivajña, Muralidhara Chaturveda (ed.), Varanasi 1981, p.326.

> the *madhyamādhikara* (the first chapter of *Siddhāntaśiromaṇi*) onwards, of the (motion of the) heavenly bodies, a mathematician will not have any value in the scholarly assemblies; without the *upapatti*-s he himself will not be free of doubt (*niḥsaṃśaya*). Since *upapatti* is clearly perceivable in the (armillary) sphere like a berry in the hand, I therefore begin the *golādhyāya* (section on spherics) to explain the *upapatti*-s.

As the commentator Nṛsiṃha Daivajña explains, 'the *phala* (object) of *upapatti* is *pāṇḍitya* (scholarship) and also removal of doubts (for oneself) which would enable one to reject wrong interpretations made by others due to *bhrānti* (confusion) or otherwise.'[28]

The same view is reiterated by Gaṇeśa Daivajña in his preface to *Buddhivilāsinī*:

> व्यक्ते वाव्यक्तसंज्ञे यदुदितमखिलं नोपपत्तिं विना तत्
> निर्भ्रान्तो वा ऋते तां सुगणकसदसि प्रौढतां नैति चायम् ।
> प्रत्यक्षं दृश्यते सा करतलकलितादर्शवत् सुप्रसन्ना
> तस्मादग्र्योपपत्तिं निगदितुमखिलम् उत्सहे बुद्धिवृद्ध्यै ॥[29]

> Whatever is stated in the *vyakta* or *avyakta* branches of mathematics, without *upapatti*, will not be rendered *nirbhrānta* (free from confusion); will not have any value in an assembly of mathematicians. The *upapatti* is directly perceivable like a mirror in hand. It is therefore, as also for the elevation of the intellect (*buddhi-vṛddhi*), that I proceed to enunciate *upapatti*-s in entirety.

Thus as per the Indian mathematical tradition, the purpose of *upapatti* is mainly (i) To remove doubts and confusion regarding the validity and interpretation of mathematical results and procedures; and, (ii) To obtain assent in the community of mathematicians.

Further, in the Indian tradition, mathematical knowledge is not taken to be different in any 'fundamental sense' from that in natural sci-

[28] *Siddhantaśiromaṇi* , cited above, p.326.
[29] *Buddhivilāsinī*, cited above, p.3.

ences. The valid means for acquiring knowledge in mathematics are the same as in other sciences: *Pratyakṣa* (perception), *Anumāna* (inference), *Śabda* or *Āgama* (authentic tradition). In his *Vāsanābhāṣya* on *Siddhāntaśiromaṇi* Bhāskarācārya refers to the sources of valid knowledge (*pramāṇa*) in mathematical astronomy, and declares that

> यद्येवमुच्यते गणितस्कन्धे उपपत्तिमान् एवागमः प्रमाणम् ।[30]
>
> For all that is discussed in Mathematical Astronomy, only an authentic tradition or established text which is supported by *upapatti* will be a *pramāṇa*.

Upapatti here includes observation. Bhāskarācārya, for instance, says that the *upapatti* for the mean periods of planets involves observations over very long periods.

3.3 The limitations of *Tarka* or proof by contradiction

An important feature that distinguishes the *upapatti*-s of Indian mathematicians is that they do not generally employ the method of proof by contradiction or *reductio ad absurdum*. Sometimes arguments, which are somewhat similar to the proof by contradiction, are employed to show the non-existence of an entity, as may be seen from the following *upapatti* given by Kṛṣṇa Daivajña to show that "a negative number has no square root":

> The square-root can be obtained only for a square. A negative number is not a square. Hence how can we consider its square-root? It might however be argued: 'Why will a negative number not be a square? Surely it is not a royal fiat'... Agreed. Let it be stated by you who claim that a negative number is a square as to whose square it is: Surely not of a positive number, for the square of a positive number is always positive by the rule...Not also of a negative number. Because then also the square will be positive by the rule...This being the case, we do not see any such number whose square becomes negative...[31]

[30] *Siddhāntaśiromaṇi*, cited above, p.30.
[31] *Bījapallava*, cited earlier, p.19.

Such arguments, known as *tarka* in Indian logic, are employed only to prove the non-existence of certain entities, but not for proving the existence of an entity, which existence is not demonstrable (at least in principle) by other direct means of verification.

In rejecting the method of indirect proof as a valid means for establishing existence of an entity which existence cannot even in principle be established through any direct means of proof, the Indian mathematicians may be seen as adopting what is nowadays referred to as the 'constructivist' approach to the issue of mathematical existence. But the Indian philosophers, logicians, etc., do much more than merely disallow certain existence proofs. The general Indian philosophical position is one of eliminating from logical discourse all reference to such *aprasiddha* entities, whose existence is not even in principle accessible to all means of verification.[32] This appears to be also the position adopted by the Indian mathematicians. It is for this reason that many an "existence theorem" (where all that is proved is that the non-existence of a hypothetical entity is incompatible with the accepted set of postulates) of Greek or modern Western mathematics would not be considered significant or even meaningful by Indian mathematicians.

3.4 *Upapatti* and "Proof"

We now summarize our discussion on the classical Indian understanding of the nature and validation of mathematical knowledge:

1. The Indian mathematicians are clear that results in mathematics, even those enunciated in authoritative texts, cannot be accepted as valid unless they are supported by *yukti* or *upapatti* . It is not enough that one has merely observed the validity of a result in a large number of instances.

2. Several commentaries written on major texts of Indian mathematics and astronomy present *upapatti*-s for the results and procedures enunciated in the text.

[32]For the approach adopted by Indian philosophers to *tarka* or the method of indirect proof see for instance, M.D.Srinivas, "The Indian Approach to Formal Logic and the Methodology of Theory Construction: A Preliminary View", PPST Bulletin **9**, 32-59, 1986.

3. The *upapatti*-s are presented in a sequence proceeding systematically from known or established results to finally arrive at the result to be established.

4. In the Indian mathematical tradition the *upapatti*-s mainly serve to remove doubts and obtain consent for the result among the community of mathematicians.

5. The *upapatti*-s may involve observation or experimentation. They also depend on the prevailing understanding of the nature of the mathematical objects involved.

6. The method of *tarka* or "proof by contradiction" is used occasionally. But there are no *upapatti*-s which purport to establish existence of any mathematical object merely on the basis of *tarka* alone.

7. The Indian mathematical tradition did not subscribe to the ideal that *upapatti*-s should seek to provide irrefutable demonstrations establishing the absolute truth of mathematical results. There was apparently no attempt to present the *upapatti*-s as a part of a deductive axiomatic system. While Indian mathematics made great strides in the invention and manipulation of symbols in representing mathematical results and in facilitating mathematical processes, there was no attempt at formalization of mathematics.

The classical Indian understanding of the nature and validation of mathematical knowledge seems to be rooted in the larger epistemological perspective developed by the *Nyāya* school of Indian logic. Some of the distinguishing features of *Nyāya* logic, which are particularly relevant in this context, are: That it is a logic of cognitions (*jñāna*) and not "propositions", that it has no concept of pure "formal validity" as distinguished from "material truth", that it does not distinguish necessary and contingent truth or analytical and synthetic truth, that it does not admit, in logical discourse, premises which are known to be false or terms that are non-instantiated, that it does not accord *tarka* or "proof by contradiction" a status of independent *pramāṇa* or means of knowledge, and so on.[33]

[33]For a discussion of some of these features, see J.N.Mohanty: *Reason and Tradition in Indian Thought*, Oxford, 1992.

The close relation between the methodology of Indian mathematics and *Nyāya* epistemology, has been commented upon by a leading scholar of *navya-nyāya*:

> The western concept of proof owes its origin to Plato's distinction between knowledge and opinion or between reason and sense. According to Plato, reason not merely knows objects having ontological reality, but also yields a knowledge which is logically superior to opinion to which the senses can aspire. On this distinction is based the distinction between contingent and necessary truths, between material truth and formal truth, between rational knowledge which can be proved and empirical knowledge which can only be verified ...
>
> As a matter of fact, the very concept of reason is unknown in Indian philosophy. In the systems which accept inference as a source of true knowledge, the difference between perception and inference is not explained by referring the two to two different faculties of the subject, sense and reason, but by showing that inferential knowledge is caused in a special way by another type of knowledge (*vyāpti-jñāna* [knowledge of invariable concomitance]), whereas perception is not so caused ...
>
> In Indian mathematics we never find a list of self-evident propositions which are regarded as the basic premises from which other truths of mathematics follow ...
>
> Euclid was guided in his axiomatization of geometry by the Aristotelean concept of science as a systematic study with a few axioms which are self-evident truths. The very concept of a system thus involves a distinction between truths which need not be proved (either because they are self-evident as Aristotle thought, or because they have been just chosen as the primitive propositions of a system as the modern logicians think) and truths which require proof. But this is not enough. What is important is to suppose that the number of self-evident truths or primitive propositions is very small and can be exhaustively enumerated.

> Now there is no Indian philosophy which holds that some truths do not require any proof while others do. The systems which accept *svataḥprāmāṇyavāda* hold that all (true) knowledge is self-evidently true, and those which accept *parataḥprāmāṇyavāda* hold that all (true) knowledge requires proof; there is no system which holds that some truths require proof while others do not ...[34]

3.5 Towards a new epistemology for Mathematics

Mathematics today, rooted as it is in the modern Western tradition, suffers from serious limitations. Firstly, there is the problem of 'foundations' posed by the ideal view of mathematical knowledge as a set of infallible eternal truths. The efforts of mathematicians and philosophers of the West to secure for mathematics the status of indubitable knowledge has not succeeded; and there is a growing feeling that this goal may turn out to be a mirage.

After surveying the changing status of mathematical truth from the Platonic position of "truth in itself", through the early twentieth century position that "mathematical truth resides ... uniquely in the logical deductions starting from premises arbitrarily set by axioms", to the twentieth century developments which question the infallibility of these logical deductions themselves, Bourbaki are forced to conclude that:

> To sum up, we believe that mathematics is destined to survive, and that the essential parts of this majestic edifice will never collapse as a result of the sudden appearance of a contradiction; but we cannot pretend that this opinion rests on anything more than experience. Some will say that this is small comfort; but already for two thousand five hundred years mathematicians have been correcting their errors to the consequent enrichment and not impoverishment of this science; and this gives them the right to face the future with serenity.[35]

[34] Sibajiban Bhattacharya, 'The Concept of Proof in Indian Mathematics and Logic', in *Doubt, Belief and Knowledge*, Delhi, 1987, p.193, 196.

[35] N.Bourbaki, *Elements of Mathematics: Theory of Sets*, Springer 1968, p.13; see also N.Bourbaki, *Elements of History of Mathematics*, Springer 1994, p.1-45.

Apart from the problems inherent in the goals set for mathematics, there are also other serious inadequacies in the Western epistemology and philosophy of mathematics. The ideal view of mathematics as a formal deductive system gives rise to serious distortions. Some scholars have argued that this view of mathematics has rendered philosophy of mathematics barren and incapable of providing any understanding of the actual history of mathematics, the logic of mathematical discovery and, in fact, the whole of creative mathematical activity.[36]

There is also the inevitable chasm between the ideal notion of infallible mathematical proof and the actual proofs that one encounters in standard mathematical practice, as portrayed in a recent book:

> On the one side, we have real mathematics, with proofs, which are established by the 'consensus of the qualified'. A real proof is not checkable by a machine, or even by any mathematician not privy to the *gestalt*, the mode of thought of the particular field of mathematics in which the proof is located. Even to the 'qualified reader' there are normally differences of opinion as to whether a real proof (i.e., one that is actually spoken or written down) is complete or correct. These doubts are resolved by communication and explanation, never by transcribing the proof into first order predicate calculus. Once a proof is 'accepted', the results of the proof are regarded as true (with very high probability). It may take generations to detect an error in a proof... On the other side, to be distinguished from real mathematics, we have 'meta-mathematics'... It portrays a structure of proofs, which are indeed infallible 'in principle'... [The philosophers of mathematics seem to claim] that the problem of fallibility in real proofs... has been conclusively settled by the presence of a notion of infallible proof in meta-mathematics... One wonders how they would justify such a claim.[37]

Apart from the fact that the modern Western epistemology of mathematics fails to give an adequate account of the history of mathematics

[36] I.Lakatos, *Proofs and Refutations: The Logic of Mathematical Discovery*, Cambridge 1976.

[37] Philip J.Davis and Reuben Hersh, *The Mathematical Experience*, Boston, 1981, p.354-5.

and standard mathematical practice, there is also the growing awareness that the ideal of mathematics as a formal deductive system has had serious consequences in the teaching of mathematics. The formal deductive format adopted in mathematics books and articles greatly hampers understanding and leaves the student with no clear idea of what is being talked about.

Notwithstanding all these critiques, it is not likely that, within the Western philosophical tradition, any radically different epistemology of mathematics will emerge; and so the driving force for modern mathematics is likely to continue to be a search for infallible eternal truths and modes of establishing them, in one form or the other. This could lead to 'progress' in mathematics, but it would be progress of a rather limited kind.

If there is a major lesson to be learnt from the historical development of mathematics, it is perhaps that the development of mathematics in the Greco-European tradition was seriously impeded by its adherence to the cannon of ideal mathematics as laid down by the Greeks. In fact, it is now clearly recognized that the development of mathematical analysis in the Western tradition became possible only when this ideal was given up during the heydays of the development of "infinitesimal calculus" during $16^{th} - 18^{th}$ centuries. As one historian of mathematics notes:

> It is somewhat paradoxical that this principal shortcoming of Greek mathematics stemmed directly from its principal virtue–the insistence on absolute logical rigour...Although the Greek bequest of deductive rigour is the distinguishing feature of modern mathematics, it is arguable that, had all the succeeding generations also refused to use real numbers and limits until they fully understood them, the calculus might never have been developed and mathematics might now be a dead and forgotten science.[38]

It is of course true that the Greek ideal has gotten reinstated at the heart of mathematics during the last two centuries, but it seems that most of the foundational problems of mathematics can also be perhaps traced to the same development. In this context, study of alternative epistemologies such as that developed in the Indian tradition of mathematics, could prove to be of great significance for the future of mathematics.

[38] C.H.Edwards, *History of Calculus*, New York 1979, p.79.

Appendices

A List of Works Containing *Upapatti*-s

The following are some of the important commentaries available in print, which present *upapatti*-s of results and procedures in mathematics and astronomy:

1. *Bhāṣya* of Bhāskara I (c.629) on *Āryabhaṭīya* of Āryabhaṭa (c.499), K.S.Shukla (ed.), New Delhi 1975.

2. *Bhāṣya* of Govindasvāmin (c.800) on *Mahābhāskarīya* of Bhāskara I (c.629), T.S.Kuppanna Sastri (ed.), Madras 1957.

3. *Vāsanābhāṣya* of Caturveda Pṛthūdakasvāmin (c.860) on *Brāhmasphuṭasiddhānta* of Brahmagupta (c.628), Chs. I-III, XXI, Ramaswarup Sharma (ed.), New Delhi 1966; Ch XXI, Edited and Translated by Setsuro Ikeyama, Ind. Jour. Hist. Sc. Vol. **38**, 2003.

4. *Vivaraṇa* of Bhāskarācārya II (c.1150) on *Śiṣyadhīvṛddhidatantra* of Lalla (c.748), Chandrabhanu Pandey (ed.), Varanasi 1981.

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6. *Mitākṣarā* or *Vāsanā* of Bhāskarācārya II (c.1150) on his own *Siddhāntaśiromaṇi*, Bapudeva Sastrin (ed.), Varanasi 1866; Muralidhara Chaturveda (ed.), Varanasi 1981.

7. *Vāsanābhāṣya* of Āmarāja (c.1200) on *Khaṇḍakhādyaka* of Brahmagupta (c.665), Babuaji Misra (ed.), Calcutta 1925.

8. *Gaṇitabhūṣaṇa* of Makkībhaṭṭa (c.1377) on *Siddhāntaśekhara* of Śrīpati (c.1039), Chs. I-III, Babuaji Misra (ed.), Calcutta 1932.

9. *Siddhāntadīpikā* of Parameśvara (c.1431) on the *Bhāṣya* of Govindasvāmin (c.800) on *Mahābhāskarīya* of Bhāskara I (c.629), T.S. Kuppanna Sastri (ed.), Madras 1957.

10. *Āryabhaṭīyabhāṣya* of Nīlakaṇṭha Somasutvan (c.1501) on *Āryabhaṭīya* of Āryabhaṭa (c.499), K. Sambasiva Sastri (ed.), 3 Vols., Trivandrum 1931, 1932, 1957.

11. *Yuktibhāṣā* (in Malayalam) of Jyeṣṭhadeva (c.1530); *Gaṇitādhyāya*, RamaVarma Thampuran and A.R. Akhileswara Aiyer (eds.), Trichur 1948; K.Chandrasekharan (ed.), Madras 1953. Edited and Translated by K.V. Sarma with Explanatory Notes by K. Ramasubramanian, M.D. Srinivas and M.S. Sriram (in Press).

12. *Yuktidīpikā* of Śaṅkara Vāriyar (c.1530) on *Tantrasaṅgraha* of Nīlakaṇṭha Somasutvan (c.1500), K.V.Sarma (ed.), Hoshiarpur 1977.

13. *Kriyākramakarī* of Śaṅkara Vāriyar (c.1535) on *Līlāvatī* of Bhāskarācārya II (c.1150), K.V.Sarma (ed.), Hoshiarpur 1975.

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16. *Ṭīkā* of Mallāri (c.1550) on *Grahalāghava* of Gaṇeśa Daivajña (c.1520), Balachandra (ed.), Varanasi 1865; Kedaradatta Joshi (ed.), Varanasi 1981.

17. *Bījanavāṅkurā* or *Bījapallavam* of Kṛṣṇa Daivajña (c.1600) on *Bījagaṇita* of Bhāskarācārya II (c.1150), V.G.Apte (ed.), Pune 1930; T.V.Radha Krishna Sastri (ed.), Tanjore 1958; Biharilal Vasistha (ed.), Jammu 1982.

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19. *Gūḍhārthaprakāśa* of Raṅganātha (c.1603) on *Sūryasiddhānta*, Jivananda Vidyasagara (ed.), Calcutta 1891; Reprint, Varanasi 1990.

20. *Vāsanāvārttika*, commentary of Nṛsiṃha Daivajña (c.1621) on *Vāsanābhāṣya* of Bhāskarācārya II, on his own *Siddhāntaśiromaṇi* (c.1150), Muralidhara Chaturveda (ed.), Varanasi 1981.

21. *Marīci* of Munīśvara (c.1630) on *Siddhantaśiromaṇi* of Bhāskarācārya (c.1150), *Madhyamādhikāra*, Muralidhara Jha (ed.), Varanasi 1908; *Grahagaṇitādhyāya*, Kedaradatta Joshi (ed.), 2 vols. Varanasi 1964; *Golādhyāya*, Kedaradatta Joshi (ed.), Delhi 1988.

22. *Āśayaprakāśa* of Munīśvara (c.1646) on his own *Siddhāntasārvabhauma*, *Gaṇitādhyāya* Chs. I-II, Muralidhara Thakura (ed.), 2 Vols, Varanasi 1932, 1935; Chs. III-IX, Mithalal Ojha (ed.), Varanasi 1978.

23. *Śeṣavāsanā* of Kamalākarabhaṭṭa (c.1658) on his own *Siddhāntatattvaviveka*, Sudhakara Dvivedi (ed.), Varanasi 1885; Reprint, Varanasi 1991.

24. *Sauravāsanā* of Kamalākarabhaṭṭa (c.1658) on *Sūryasiddhānta*, Chs. I-X, Sri Chandra Pandeya (ed.), Varanasi 1991.

25. *Gaṇitayuktayaḥ*, Tracts on Rationale in Mathematical Astronomy by Various Kerala Astronomers (c.$16^{th}-19^{th}$ century), K.V.Sarma (ed.), Hoshiarpur 1979.

B Some *Upapatti*-s from *Yuktibhāṣā* (c.1530)

In this Appendix we shall present some of the proofs contained in the Mathematics part of the celebrated Malayalam text *Yuktibhāṣā* [39] of Jyeṣṭhadeva (c.1530). This part is divided into seven chapters, of which the last two, entitled *Paridhi* and *Vyāsa* (Circumference and Diameter) and *Jyānayana* (Computation of Sines), contain many important results concerning infinite series and fast convergent approximations

[39] *Yuktibhāṣā* (in Malayalam) of Jyeṣṭhadeva (c.1530); *Gaṇitādhyāya*, Ramavarma Thampuran and A.R. Akhileswara Aiyer (eds.), Trichur 1947; K. Chandrasekharan (ed.), Madras 1953; Edited, along with an ancient Sanskrit version *Gaṇitayuktibhāṣā* and English Translation, by K.V.Sarma, with Explanatory Notes by K.Ramasubramanian, M.D.Srinivas and M.S.Sriram (in press).

for π and the trigonometric functions. In the preamble to his work, Jyeṣṭhadeva states that his work gives an exposition of the mathematics necessary for the computation of planetary motions as expounded in *Tantrasaṅgraha* of Nīlakaṇṭha (c.1500). The proofs given in *Yuktibhāṣā* have been reproduced (mostly in the form of Sanskrit verses or *kārikās*) by Śaṅkara Vāriyar in his commentaries *Yuktidīpikā* [40] on *Tantrasaṅgraha* and *Kriyākramakarī* [41] on *Līlāvatī*. Since the later work is considered to be written around 1535 A.D., the time of composition of *Yuktibhāṣā* may reasonably be placed around 1530 A.D.

In what follows, we shall present a brief outline of some of the mathematical topics and proofs given in Chapters VI and VII of *Yuktibhāṣā*, following closely the order which they appear in the text.

B.1 Chapter VI : *Paridhi and Vyāsa* (Circumferene and Diameter)

The chapter starts with a proof of *bhujā-koṭi-karṇa-nyāya* (the so called Pythagoras theorem), which has also been proved earlier in the first chapter of the work.[42] It is then followed by a discussion of how to arrive at successive approximations to the circumference of a circle by giving a systematic procedure for computing successively the perimeters of circumscribing square, octagon, regular polygon of sides 16, 32, and so on. The treatment of infinite series expansions is taken up thereafter.

B.1.1 To obtain the circumference without calculating square-roots

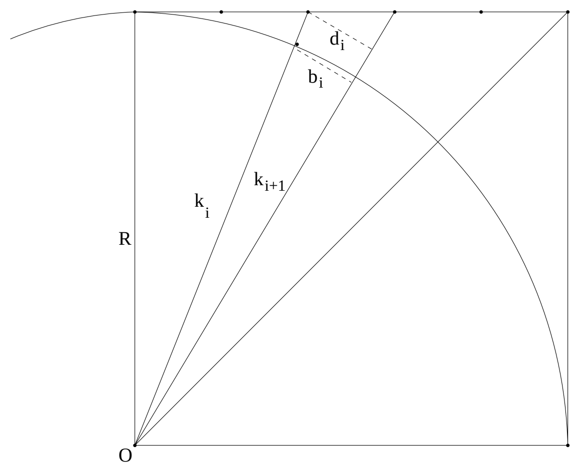
Consider a quadrant of the circle, inscribed in a square and divide a side of the square, which is tangent to the circle, into a large number of equal

[40] *Yuktidīpikā* of Sankara Variyar (c.1530) on *Tantrasaṅgraha* of Nīlakaṇṭha Somasutvan (c.1500), K.V.Sarma (ed.), Hoshiarpur 1977. At the end of each chapter of this work, Śaṅkara states that he is only presenting the material which has been well expounded by the great *dvija* of the Parakroḍha house, Jyeṣṭhadeva.

[41] *Kriyākramakarī* of Śaṅkara Vāriyar (c.1535) on *Līlāvatī* of Bhāskarācārya II (c.1150), K.V.Sarma (ed.), Hoshiarpur 1975.

[42] In fact, according to *Yuktibhāṣā* , almost all mathematical computations are pervaded (*vyāpta*) by the *trairāśika-nyāya* (the rule of proportion as exemplified for instance in the case of similar triangles)and the *bhujā-koṭi-karṇa-nyāya*.

parts. The more the number of divisions the better is the approximation to the circumference.



$C/8$ (one eighth of the circumference) is approximated by the sum of the *jyārdhās* (half-chords) b_i of the arc-bits to which the circle is divided by the *karṇas* (hypotenuses) which join the points which the divide tangent to the centre of the circle. Let k_i be the length of the i^{th} *karṇa*. Then,

$$b_i = \left(\frac{R}{k_i}\right) d_i = \frac{R}{k_i}\left[\left(\frac{R}{n}\right)\frac{R}{k_{i+1}}\right] = \left(\frac{R}{n}\right)\frac{R^2}{k_i k_{i+1}}$$

Hence

$$\frac{\pi}{4} = \frac{C}{8R} = \left(\frac{1}{n}\right)\sum_{i=0}^{n-1}\frac{R^2}{k_i k_{i+1}} \approx \left(\frac{1}{n}\right)\sum_{i=0}^{n-1}\left(\frac{R^2}{k_i}\right)^2$$
$$= \left(\frac{1}{n}\right)\sum_{i=0}^{n-1}\frac{R^2}{\left[R^2 + i^2\left(\frac{R}{n}\right)^2\right]}$$

Series expansion of each term in the RHS is obtained by iterating the relation

$$\frac{a}{b} = \frac{a}{c} - \left(\frac{a}{b}\right)\left(\frac{b-c}{c}\right),$$

which leads to

$$\frac{a}{b} = \frac{a}{c} - \left(\frac{a}{b}\right)\left(\frac{b-c}{c}\right) + \left(\frac{a}{c}\right)\left(\frac{b-c}{c}\right)^2 + \ldots.$$

This (binomial) expansion is also justified later by showing how the partial sums in the following series converge to the result.

$$\frac{100}{10} = \frac{100}{8} - \left(\frac{100}{10}\right)\left(\frac{10-8}{8}\right) + \left(\frac{100}{8}\right)\left(\frac{10-8}{8}\right)^2 - \ldots.$$

Thus

$$\frac{\pi}{4} = 1 - \left(\frac{1}{n}\right)^3 \sum_{i=1}^{n} i^2 + \left(\frac{1}{n}\right)^5 \sum_{i=1}^{n} i^4 - \ldots$$

When n becomes very large, this leads to the series given in the rule of Mādhava *vyāse vāridhinihate* …[43]

$$\frac{C}{4D} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$$

B.1.2 *Samaghāta-saṅkalita* – Sums of powers of natural numbers

In the above derivation, the following estimate was employed for the *samaghāta-saṅkalita* of order k, for large n :

$$S_n^{(k)} = 1^k + 2^k + 3^k + \ldots + n^k \approx \frac{n^{k+1}}{(k+1)}$$

This is proved first for the case of *mūla-saṅkalita*

$$\begin{aligned} S_n^{(1)} &= 1 + 2 + 3 + \ldots + n \\ &= [n-(n-1)] + [n-(n-2)] + \cdots + n \\ &= n^2 - S_{n-1}^{(1)} \end{aligned}$$

Hence for large n,

$$S_n^{(1)} \approx \frac{n^2}{2}$$

Then, for the *varga-saṅkalita* and the *ghana-saṅkalita*, the following estimates are proved for large n:

$$\begin{aligned} S_n^{(2)} &= 1^2 + 2^2 + 3^2 + \ldots + n^2 \approx \frac{n^3}{3} \\ S_n^{(3)} &= 1^3 + 2^3 + 3^3 + \ldots + n^3 \approx \frac{n^4}{4} \end{aligned}$$

[43] This result is attributed to Mādhava by Śaṅkara Vāriyar in *Kriyākramakarī*, cited earlier, p.379; see also *Yuktidīpikā* , cited earlier, p.101.

It is then observed that, in each case, the derivation above is based on the result

$$nS_n^{(k-1)} - S_n^{(k)} = S_{n-1}^{(k-1)} + S_{n-2}^{(k-1)} + \ldots + S_1^{(k-1)}$$

It is observed that the right hand side of the above equation is a repeated sum of the lower order $(k-1)$ *saṅkalita*. Now if we have already estimated this lower order *saṅkalita*, $S_n^{(k-1)} \approx \frac{n^k}{k}$, then

$$\begin{aligned} nS_n^{(k-1)} - S_n^{(k)} &\approx \frac{(n-1)^k}{k} + \frac{(n-2)^k}{k} + \frac{(n-3)^k}{k} + \ldots \\ &\approx \left(\frac{1}{k}\right) S_n^{(k)}. \end{aligned}$$

Hence, for the general *samaghāta-saṅkalita*, we obtain the estimate

$$S_n^{(k)} \approx \frac{n^{k+1}}{(k+1)}.$$

B.1.3 *Vāra-saṅkalita* – Repeated summations

The *vāra-saṅkalita* or *saṅkalita-saṅkalita* or repeated sums, are defined as follows:

$$\begin{aligned} V_n^{(1)} &= S_n^{(1)} = 1 + 2 + \ldots + (n-1) + n \\ V_n^{(r)} &= V_1^{(r-1)} + V_2^{(r-1)} + \ldots + V_n^{(r-1)} \end{aligned}$$

It is shown that, for large n

$$V_n^{(r)} \approx \frac{n^{r+1}}{(r+1)!}.$$

B.1.4 *Cāpīkaraṇa* – Determination of the arc

This can be done by the series given by the rule[44] *iṣṭajyātrijyayorghātāt* ... which is derived in the same way as the above series for $\frac{C}{8}$.

$$R\theta = R\left(\frac{\sin\theta}{\cos\theta}\right) - \frac{R}{3}\left(\frac{\sin\theta}{\cos\theta}\right)^3 + \frac{R}{5}\left(\frac{\sin\theta}{\cos\theta}\right)^5 - \ldots$$

[44] See for instance, *Kriyākramakarī*, cited earlier, p.95-96.

It is noted that $|\frac{\sin\theta}{\cos\theta}| \leq 1$, is a necessary condition for the terms in the above series to progressively lead to the result. Using the above, for $\theta = \frac{\pi}{6}$, the following series is obtained:

$$C = \sqrt{12D^2}\left[1 - \frac{1}{(3.3)} + \frac{1}{(3^2.5)} - \frac{1}{(3^3.7)} + \ldots\right].$$

B.1.5 *Antya-saṃskāra* – Correction term to obtain accurate circumference

Let us set

$$\frac{C}{4D} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^{n-1}\frac{1}{(2n-1)} + (-1)^n\frac{1}{a_n}.$$

Then the *saṃskāra-hāraka* (correction divisor), a_n will be accurate if

$$\frac{1}{a_n} + \frac{1}{a_{n+1}} = \frac{1}{2n+1}.$$

This leads to the successive approximations:[45]

$$\begin{aligned}
\frac{\pi}{4} &\approx 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^{n-1}\frac{1}{(2n-1)} + (-1)^n\frac{1}{4n}, \\
\frac{\pi}{4} &\approx 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^{n-1}\frac{1}{(2n-1)} + (-1)^n\frac{1}{4n + \frac{4}{4n}}, \\
&= 1 - \frac{1}{3} + \frac{1}{5} - \ldots + (-1)^{n-1}\frac{1}{(2n-1)} + (-1)^n\frac{n}{(4n^2+1)}.
\end{aligned}$$

Later at the end of the chapter, the rule[46] *ante samasaṅkhyādalavargaḥ* . . . , is cited as the *sūkṣmatara-saṃskāra*, a much more accurate correction:

$$\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{(-1)^{n-1}}{(2n-1)} + \frac{(-1)^n(n^2+1)}{(4n^3+5n)},$$

[45] These are attributed to Mādhava in *Kriyākramakarī*, cited earlier, p.279; also cited in *Yuktidīpikā* , cited earlier, p.101.

[46] *Kriyākramakarī*, cited earlier, p.390; *Yuktidīpikā* , cited earlier, p.103.

B.1.6 Transformation of series

The above correction terms can be used to transform the series for the circumference as follows:

$$\frac{C}{4D} = \frac{\pi}{4} = \left[1 - \frac{1}{a_1}\right] - \left[\frac{1}{3} - \frac{1}{a_1} - \frac{1}{a_2}\right] + \left[\frac{1}{5} - \frac{1}{a_2} - \frac{1}{a_3}\right] \ldots$$

It is shown that, using the second order correction terms, we obtain the following series given by the rule[47] *samapañcāhatayoḥ ...*

$$\frac{C}{16D} = \frac{1}{(1^5 + 4.1)} - \frac{1}{(3^5 + 4.3)} + \frac{1}{(5^5 + 4.5)} - \ldots$$

It is also noted that by using merely the lowest order correction terms, we obtain the following series given by the rule[48] *vyāsād vāridhinihatāt* ...

$$\frac{C}{4D} = \frac{3}{4} + \frac{1}{(3^3 - 3)} - \frac{1}{(5^3 - 5)} + \frac{1}{(7^3 - 7)} - \ldots$$

B.1.7 Other series expansions

It is further noted, by using non-optimal correction divisors in the above transformed series, one can also obtain the following results given in the rules[49] *dvyādiyujāṃ vā kṛtayo* ... and *dvyādeścaturādervā ...*

$$\begin{aligned}
\frac{C}{4D} &= \frac{1}{2} + \frac{1}{(2^2 - 1)} - \frac{1}{(4^2 - 1)} + \frac{1}{(6^2 - 1)} - \ldots \\
\frac{C}{8D} &= \frac{1}{2} - \frac{1}{(4^2 - 1)} - \frac{1}{(8^2 - 1)} - \frac{1}{(12^2 - 1)} - \ldots \\
\frac{C}{8D} &= \frac{1}{(2^2 - 1)} + \frac{1}{(6^2 - 1)} + \frac{1}{(10^2 - 1)} - \ldots
\end{aligned}$$

[47] *Kriyākramakarī*, cited earlier, p.390; *Yuktidīpikā* , cited earlier, p.102.
[48] *Kriyākramakarī*, cited earlier, p.390; *Yuktidīpikā* , cited earlier, p.102.
[49] *Kriyākramakarī*, cited earlier, p.390; *Yuktidīpikā* , cited earlier, p.103.

B.2 Chapter VII : *Jyānayanam* – Computation of Sines

B.2.1 *Jyā, koti* and *śara* – $R\sin x,\ R\cos x$ and $R(1-\cos x)$

First is discussed the construction of an inscribed regular hexagon with side equal to the radius, which gives the value of $R\sin(\frac{\pi}{6})$. Then are derived the relations:

$$R\sin\left(\frac{\pi}{2}-x\right) = R\cos x = R(1-\text{versin } x)$$
$$R\sin\left(\frac{x}{2}\right) = \frac{1}{2}[(R\sin x)^2+(R\text{ versin } x)^2]^{\frac{1}{2}}.$$

Using the above relations several Rsines can be calculated starting from the following:

$$R\sin\left(\frac{\pi}{6}\right) = \frac{R}{2}$$
$$R\sin\left(\frac{\pi}{2}\right) = \left(\frac{R^2}{2}\right)^{\frac{1}{2}}.$$

This is one way of determining the *paṭhita-jyā* (enunciated or tabulated sine values), when a quadrant of a circle is divided into 24 equal parts of $3°45' = 225'$ each. To find the Rsines of intermediate values, a first approximation is

$$R\sin(x+h) \approx R\sin x + h\ R\cos x.$$

Then is derived the following better approximation as given in the rule[50] *iṣṭadoḥkoṭidhanuṣoḥ* ...:

$$R\sin(x+h) \approx R\sin x + \left(\frac{2}{\Delta}\right)\left(R\cos x - \left(\frac{1}{\Delta}\right)R\sin x\right)$$
$$R\cos(x+h) \approx R\cos x + \left(\frac{2}{\Delta}\right)\left(R\sin x - \left(\frac{1}{\Delta}\right)R\cos x\right),$$

where $\Delta = \frac{2R}{h}$.

[50] *Tantrasaṅgraha* , 2.10-14.

B.2.2 Accurate determination of sines

Given an arc $s = Rx$, divide it into n equal parts and let the *piṇḍa-jyās* B_j, and *śaras* $S_{j-\frac{1}{2}}$, with $j = 0, 1, \ldots n$, be given by

$$\begin{aligned} B_j &= R \sin\left(\frac{jx}{n}\right), \\ S_{j-\frac{1}{2}} &= R \text{ vers } \left[\frac{(j-\frac{1}{2})x}{n}\right] \end{aligned}$$

If α be the *samasta-jyā* (total chord) of the arc $\frac{s}{n}$, then the second order sine difference (*jyā-khaṇḍāntara*) is shown to satisfy

$$\begin{aligned} (B_{j+1} - B_j) - (B_j - B_{j-1}) &= \left(\frac{\alpha}{r}\right)(S_{j-\frac{1}{2}} - S_{j+\frac{1}{2}}) \\ &= \left(\frac{\alpha}{r}\right)^2 B_j, \end{aligned}$$

for $j = 1, 2, \ldots n$. From this are derived the relations

$$\begin{aligned} S_{n-\frac{1}{2}} - S_{\frac{1}{2}} &= \left(\frac{\alpha}{r}\right)(B_1 + B_2 + \ldots + B_{n-1}), \\ B_n - nB_1 &= -\left(\frac{\alpha}{R}\right)^2 [B_1 + (B_1 + B_2) + \ldots + (B_1 + \ldots + B_{n-1})] \\ &= -\left(\frac{\alpha}{r}\right)\left[S_{\frac{1}{2}} + S_{\frac{3}{2}} + \cdots + S_{n-\frac{1}{2}} - nS_{\frac{1}{2}}\right] \end{aligned}$$

If B and S are the *jyā* and *śara* of the arc s, then it is noted that, in the limit of very large n, we have as a first approximation

$$B_n \approx B, \;\; B_j \approx \frac{js}{n}, \;\; S_{n-\frac{1}{2}} \approx S, S_{\frac{1}{2}} \approx 0 \;\text{ and }\; \alpha \approx \frac{s}{n}.$$

Hence

$$S \approx \left(\frac{1}{R}\right)\left(\frac{s}{n}\right)^2 [1 + 2 + \ldots + n - 1] \approx \frac{s^2}{2R}.$$

and

$$\begin{aligned} B &\approx s - \left(\frac{1}{R}\right)^2 \left(\frac{s}{n}\right)^3 [1 + (1+2) + \ldots + (1 + 2 + \ldots + n - 1))] \\ &\approx s - \frac{s^3}{6R^2}. \end{aligned}$$

Iterating these results we get successive approximations for the difference between the Rsine and the arc (*jyā-cāpāntara*), leading to the following series given by the rule[51] *nihatya cāpavargeṇa* . . . :

$$R\sin\left(\frac{s}{R}\right) = B = R\left[\left(\frac{s}{R}\right) - \frac{\left(\frac{s}{R}\right)^3}{3!} + \frac{\left(\frac{s}{R}\right)^5}{5!} - \ldots\right]$$

$$R - R\cos\left(\frac{s}{R}\right) = S = R\left[\left(\frac{s}{R}\right)^2 - \frac{\left(\frac{s}{R}\right)^4}{4!} + \frac{\left(\frac{s}{R}\right)^6}{6!} - \ldots\right]$$

While carrying successive approximations, the following result for *vāra-saṅkalitas* (repeated summations) is used:

$$\sum_{j=1}^{n} \frac{j(j+1)\ldots(j+k-1)}{k!} = \frac{n(n+1)(n+2)\ldots(n+k)}{(k+1)!} \approx \frac{n^{k+1}}{(k+1)!}.$$

Then is obtained a series for the square of sine, as given by the rule[52] *nihatya cāpavargeṇa* . . .

$$\sin^2 x = x^2 - \frac{x^4}{\left(2^2 - \frac{2}{2}\right)} + \frac{x^6}{\left(2^2 - \frac{2}{2}\right)\left(3^2 - \frac{3}{2}\right)} - \ldots$$

Chapter VII of *Yuktibhāṣā* goes on to discuss different ways of deriving the *jīve-paraspara-nyāya*[53], which is followed by a detailed discussion of the cyclic quadrilateral. The chapter concludes with a derivation of the surface area and volume of a sphere.

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[51] *Yuktidīpikā* , cited earlier, p.118

[52] *Yuktidīpikā* , cited earlier, p.119.

[53] The relation between the sine and cosine of the sum or difference of two arcs with the sines and cosines of the arcs.