THE ALGORITHMIC APPROACH OF INDIAN MATHEMATICS

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The word algorithm, which is commonly used for any systematic procedure of computation, has an interesting history. It derives from the medieval word "algorism", which referred to the process of doing arithmetic by means of the Indian numerals (the so called "Hindu-Arabic numerals") following the Indian methods of calculation based on the decimal place value system. The word "algorism" itself is a corruption of the name of the Central Asian mathematician al Khwarizmi (c.825) whose book on the "Indian method of reckoning" (*Hisab al Hind*) was the source from which the Indian methods of calculation reached the Western world. The "algorists" in medieval Europe, who computed by "algorism", were at a great advantage compared to those who used the abacus or any other system of numeration such as the Roman system of numerals. The situation has been aptly described by the renowned 18th century French mathematician Pierre Simon de Laplace as follows:

> "It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity and the great ease which it has lent to all computations put our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity."

The term "Mathematics" is derived from the Greek word "Mathema" which means knowledge or learning. The Indian term for this discipline is *Ganita* which means calculation or computation. Indian Mathematics, *Ganita*, is quintessentially a science of computation. The Indian mathematical texts are not just a collection of propositions or theorems about mathematical entities, they are more in the nature of compendia of systematic and efficient procedures for computation (with numbers, geometrical figures, algebraic symbols standing for a class of mathematical objects and so on) as applicable to diverse problems. Thus a majority of the *sutras* or verses of a classical Indian mathematical text are in the form of prescriptions or rules − they are referred to by the traditional commentators as *vidhi*, *prakriya* or *karana-sutras* − rules that characterise systematic procedures.

This approach is not something special to mathematics alone, but is common to most of the Indian knowledge systems, *Sastras*. The canonical texts of different disciplines in

Indian tradition present rules, which are generally called as *sutras* or ***lakshanas*.** Most of these rules serve to characterize systematic procedures (referred to variously as *vidhi*, *kriya* or *prakriya*, *sadhana, karma* or *parikarma*, *karana,* etc.) which are designed to accomplish specific ends. In this way the Indian *Sastras* are always rooted in *vyavahara* or practical applications.

This approach of Indian *Sastras* allows them to have a great degree of flexibility in devising multiple approaches to the solutions of problems, and not get bogged down by any dogma of inviolability of the fundamental truths posited or derived in any specific theoretical formulation employed in the discipline concerned. While the canonical texts of Indian *Sastras* clearly assert the validity and the efficacy of the various procedures enunciated in them, they also simultaneously emphasise that these procedures are only *upayas*, or means for accomplishing specific ends, and there are no other restrictions which need to be imposed on them. The texts also declare that one is free to take recourse to any other set of systematic procedures, if they are equally efficacious in accomplishing the given ends.

It is the algorithmic approach which distinguishes the ancient Indian texts of Geometry, the *Sulvasutras* (prior to 800 BCE), which deal with the construction of *yajna-vedis* (altars). While the *Sulvasutras* do contain the earliest available statement of the so called Pythagoras theorem − they state it in the form "the square made by the diagonal of a rectangle is equal to the sum of the squares made by its sides" − the main purpose of the *Sulvasutras* is to describe systematic procedures for constructing and transforming geometrical figures using a *rajju* (rope) and *sanku* (pole).

It has been remarked that the Indian scientific tradition has been profoundly influenced by the methodology of the *Ashtadhyayi* of Panini (prior to 500 BCE), in the same way the Greco-European tradition is said to have been influenced by the methodology of the *Elements* of Euclid. The *Ashtadhyayi* of Panini is a truly generative grammar in the modern sense that it is a collection of about 4,000 rules using which it is possible to derive every valid utterance of the Sanskrit language starting from a collection of primitive elements such as *dhatus* (verb-roots), *pratyayas* (suffixes), etc. Panini does not present any set of propositions or truths about language, but gives us an expert system so to say, which enables us to derive (and also analyse) all the valid utterances of Sanskrit language, an achievement which is as yet unparalleled in the grammatical tradition of any of the world languages. Many of the techniques used in Panini's grammar − his abstract symbolism, use of the zero-morpheme (*lopa*), ideas of rule ordering, recursion etc. − seem to have had significant impact on the development of Indian mathematics. They have also influenced developments in modern linguistics and computer science.

The ancient text of Sanskrit prosody, the *Chandahsastra* of Pingala (c. 300 BCE) presents algorithms for converting a number to its binary form and *vice versa*. In Sanskrit prosody, any *pada* (line or foot) of a verse is analysed as a sequence of *guru* (long) and *laghu* (short) syllables, so that Pingala could essentially characterise it as a binary sequence. Pingala also gives an efficient algorithm for finding the *n*-th power of a number, which involves only around $\log_2(n)$ operations of squaring and multiplications (in contrast to the standard method which involves *n* multiplications), and was therefore adopted by all the later Indian mathematicians. Pingala's work also contains a cryptic *sutra*, which has been explained by later commentators, such as Halayudha (c. 900 CE), as a rule for the computation of binomial coefficients using a tabular form *Meru*, which is a version of the famous Pascal triangle. Pingala's work set the stage for subsequent developments in combinatorics, which were initiated in texts of prosody and music, and were formulated in a general mathematical setting by later mathematicians, Mahaviracharya (c. 850), Bhaskaracharya II (b. 1114) and especially Narayana Pandita (c. 1356)

In ancient times, *Ganita* formed an important part of the science of Astronomy (*Jyotisha*). The *Aryabhatiya* (c.499 CE) of Aryabhata, is a great classical work which summarised the entire subject of Mathematical Astronomy in 121 aphoristic verses, of which the section on Mathematics, *Ganitapada*, comprised of just 32 verses. We can see that, by that time, the Indian mathematicians had systematised most of the basic procedures of arithmetic (such as place value system, the standard algorithms for square-roots and cube-roots), algebra (solution of linear and quadratic equations), geometry (standard properties of planar and solid figures), commercial mathematics (rule of three, calculation of interest) and trigonometry, that are generally taught in schools to-day − and many more that are more advanced (such as the *kuttaka* method of solving linear indeterminate equations, computation of sine-tables, etc.) which are of importance in Astronomy.

Several detailed commentaries (*bhashyas*) were written on the cryptic verses of *Aryabhatiya*, of which the most important ones are those of Bhaskara I (c. 629 CE) and the great Kerala Astronomer Nilakantha Somayaji (1444-1544). The commentary of Bhaskara I provides detailed explanations (along with examples) for the various results and procedures given in *Aryabhatiya*. The *Aryabhatiya-bhashya* of Nilakantha presents detailed demonstrations (*upapatti, yukti*). Occasionally, Nilakantha also discusses some important refinements or modifications. We may cite for instance Nilakantha's discussion of the more accurate table of sines (due to Madhava), and more importantly his famous dictum, based on the latitudinal motion of the planets Mercury and Venus, that: "The earth is not circumscribed by their orbits [the orbits of Mercury and Venus], the earth is always outside of them". This led Nilakantha to formulate a modified planetary model according to which the five planets Mercury, Venus, Mars, Jupiter and Saturn go around the mean Sun, which in turn goes around the Earth. This was nearly a hundred years prior to a similar model being proposed by Tycho Brahe in Europe.

We shall not go into the contributions of the long tradition of illustrious astronomers and mathematicians who followed Aryabhata − and the tradition continued to flourish till the nineteenth century. We instead present some illustrations to show how the algorithmic approach of the Indian mathematicians led them to discover optimal and efficacious algorithms for diverse problems. The most famous example is of course the *Chakravala* algorithm for the solution of the quadratic indeterminate equation (the so called Pell's equation):

$$X^2 - DY^2 = 1.$$

Here D is a given positive integer which is not a square, and the problem is to solve for X, Y in integers. This problem (called *Vargaprakriti*) was first explicitly posed by Brahmagupta in his *Brahmasphutasiddhanta* (c. 628 CE), though the ancient *Sulvasutras* seem to have used the solution X=577, Y=408 for the case D=2, to get the rational approximation 577/408 for the square-root of 2. Brahmagupta also gave a rule of composition (called *bhavana*) which allows one to obtain an infinite number of solutions once a particular solution is found.

The *Chakravala* method for solving the above equation has been presented in the famous textbook of algebra, *Bijaganita*, of Bhaskaracharya (b. 1114), though it is now known that the algorithm also appears in an earlier work due to Acharya Jayadeva (prior to c. 1050). Bhaskara used this method to solve the equation

$$X^2 - 61Y^2 = 1,$$

and showed that the smallest solution is given by X=1766319049 and Y=226153980. What is intriguing is that the same example was posed as a challenge by the famous French mathematician Pierre de Fermat in February 1657 to his colleagues in France. He later on posed this and other *Vargaprakriti* equations (with different values of D) as a challenge to the British mathematicians. To cut the story short, the British mathematicians Wallace and Brouncker did come up with a method of solution, which was later systematised as an algorithm, based on the so called regular continued fraction development of the square-root of D, by Euler and Lagrange in the 1770s. In 1929, A.A.Krishnaswamy Ayyangar showed that the *Chakravala* algorithm corresponds to a so called semi-regular continued fraction expansion, and is also optimal in the sense that it takes much fewer steps to arrive at the solution than the Euler-Lagrange method. It is now known that on the average the Euler-Lagrange method takes about 40% more number of steps than the *Chakravala*.

Finally, we make a brief mention of the infinite series for Pi (the ratio of the circumference to the diameter of a circle) discovered by Sangamagrama Madhava (c. 1380-1460), the founder of the Kerala School of Astronomy. For instance, Madhava presents the following series (the so called Gregory-Leibniz series rediscovered in 1670s)

$$Pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots$$

However, Madhava is not content with merely enunciating this elegant result, as it is not of any use in actually calculating the value of Pi. Summing say fifty terms in this series does not give a value of Pi accurate even to two decimal places. The famous verses of Madhava which present the above series also go on to give a set of end-correction terms which can be used to obtain better approximations. Using only fifty terms of the above series, with the accurate end-correction term of Madhava, leads to a value of Pi accurate to 11 decimal places. Madhava also used these correction terms to transform the above series into more rapidly convergent versions. Systematic proofs of all the infinite series discovered by Madhava and their transformations may be found in the famous Malayalam work *Ganitayuktibhasha* (c.1530) of Jyeshthadeva.

The great astronomer Nilakantha was a third generation disciple of Madhava and the tradition of Kerala School continued (albeit at a modest level due to the greatly disturbed political situation of Kerala after the 1550s) till early nineteenth century. However, a century later, the algorithmic approach of Indian mathematics was in evidence again in the work of another great mathematician Srinivasa Ramanujan (1888-1920), who seems to have been a worthy successor of Madhava in his extraordinary felicity to work with infinite series, and their transformations.

Reference: An overview of Indian mathematics can be had from the NPTEL Course of 40 lectures on “Mathematics in India: From Vedic Period to Modern Times” by K.Ramasubramanian, M.S.Sriram and M.D. Srinivas, which can be accessed at: http://nptel.ac.in/courses/111101080/