Appendix F
The traditional Indian planetary model and its revision by Nīlakaṇṭha Somayājī [1]

It is now generally recognized that the Kerala school of Indian astronomers,[2] starting from Mādhava of Saṅgamagrāma (1340–1420 CE), made important contributions to mathematical analysis much before this subject developed in Europe. The Kerala astronomers derived infinite series for π, sine and cosine functions and also developed fast convergent approximations to them.[3]

Here we shall explain how the Kerala school also made equally significant discoveries in astronomy, and particularly in planetary theory. Mādhava's disciple Parameśvara of Vaṭaśśeri (c. 1380–1460) is reputed to have made continuous and careful observations for over 55 years. He is famous as the originator of the *Dṛg-gaṇita* system, which replaced the older *Parahita* system. He also discussed the geometrical picture of planetary motion as would follow from the traditional Indian planetary model.

---

[1] This appendix, prepared by K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, is a revised and updated version of the following earlier studies on the subject: (i) K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, Modification of the Earlier Indian Planetary Theory by the Kerala Astronomers (c. 1500) and the implied Heliocentric Picture of Planetary Motion, *Current Science* 66, 784–790, 1994. (ii) M. S. Sriram, K. Ramasubramanian and M. D. Srinivas (eds), *500 Years of Tantrasangraha: A Landmark in the History of Astronomy*, IIAS, Shimla 2002, pp. 29–102. (iii) Epilogue: Revision of Indian Planetary Model by Nīlakaṇṭha Somayājī, in *Gaṇita-yukti-bhāṣā* of Jyeṣṭhadeva, ed. and tr. K. V. Sarma with Explanatory Notes by K. Ramasubramanian, M. D. Srinivas and M. S. Sriram, 2 vols, Hindustan Book Agency, Delhi 2008; repr. Springer, 2009, vol II, pp. 837–856.

[2] For the Kerala school of astronomy, see for instance, K. V. Sarma, *A Bibliography of Kerala and Kerala-based Astronomy and Astrology*, Hoshiarpur 1972; K. V. Sarma, *A History of the Kerala School of Hindu Astronomy*, Hoshiarpur 1972.

[3] For overviews of the Kerala tradition of mathematics, see S. Parameswaran, *The Golden Age of Indian Mathematics*, Kochi 1998; G. G. Joseph, *The Crest of the Peacock: Non-European Roots of Mathematics*, 2nd edn. Princeton 2000; C. K. Raju, *Cultural Foundations of Mathematics: The Nature of Mathematical Proof and the Transmission of the Calculus from India to Europe in the 16th c. CE*, Pearson Education, Delhi 2007; Kim Plofker, *History of Mathematics in India: From 500 BCE to 1800 CE*, Princeton 2009; G. G. Joseph (ed.), *Kerala Mathematics: History and Possible Transmission to Europe*, B. R. Publishing, New Delhi 2009. See also the detailed mathematical notes in *Gaṇita-yukti-bhāṣā* cited above.

Nīlakaṇṭha Somayājī of Tṛkkaṇṭiyūr (c. 1444–1550), a disciple of Parameśvara's son Dāmodara, carried out a fundamental revision of the traditional planetary theory. In his treatise *Tantrasaṅgraha*, composed in 1500, Nīlakaṇṭha outlines the detailed computational scheme of his revised planetary model. For the first time in the history of Astronomy, Nīlakaṇṭha proposed that in the case of an interior planet (Mercury or Venus), the *manda*-correction or the equation of centre should be applied to what was traditionally identified as the *śighrocca* of the planet—which, in the case of interior planets, corresponds to what we currently refer to as the mean heliocentric planet. This was a radical departure from the traditional Indian planetary model where the *manda*-correction for an interior planet was applied to the mean Sun.[4]

In this way, Nīlakaṇṭha arrived at a much better formulation of the equation of centre and the latitudinal motion of the interior planets than was available either in the earlier Indian works or in the Islamic or the Greco-European traditions of astronomy till the work of Kepler, which was to come more than a hundred years later. In fact, in so far as the computation of the planetary longitudes and latitudes is concerned, Nīlakaṇṭha's revised planetary model closely approximates to the Keplerian model, except that Nīlakaṇṭha conceives of the planets as going in eccentric orbits around the mean Sun rather than the true Sun.

In his *Āryabhaṭīya-bhāṣya*, Nīlakaṇṭha explains the rationale behind his revision of the traditional planetary theory. This has to do with the fact (which was noticed by several Indian astronomers prior to Nīlakaṇṭha) that the traditional Indian planetary model employed entirely different schemes for computing the latitudes of the exterior and the interior planets. While the latitudes of the exterior planets were computed from their so-called *manda-sphuṭa* (which corresponds to what we currently refer to as the true heliocentric planet), the latitudes of the interior planets were computed from their so-called *śīghrocca*. Nīlakaṇṭha argued that since the latitude should be dependent on the deflection (from the ecliptic) of the planet itself and not of any other body, what was traditionally referred to as the *śīghrocca* of an interior planet should be identified with the planet itself. Nīlakaṇṭha also showed that this would lead to a unified treatment of the latitudinal motion of all the planets—interior as well as exterior.[5]

In *Āryabhaṭīya-bhāṣya*, Nīlakaṇṭha also discusses the geometrical picture of planetary motion implied by his revised model.[6] This geometrical picture, which is also stated by Nīlakaṇṭha succinctly in terms of a few verses in *Golasāra* and *Siddhānta-darpaṇa*, is essentially that the planets move in eccentric orbits (which

[4] It had also been a general feature of all ancient planetary theories in the Greco-European and the Islamic traditions of astronomy, till the work of Kepler, that the equation of centre for an interior planet was wrongly applied to the mean Sun.

[5] In fact, it has been noted in a later text, *Vikṣepagolavāsanā*, that Nīlakaṇṭha pioneered a revision of the traditional planetary theory in order to arrive at a unified formulation of the motion in latitude of both the interior and the exterior planets.

[6] The renowned Malayalam work *Gaṇita-yukti-bhāṣā* (c. 1530) of Jyeṣṭhadeva also gives a detailed exposition of the geometrical picture of planetary motion as per the planetary model of Nīlakaṇṭha outlined in *Tantrasaṅgraha*.

are inclined to the ecliptic) around the *śīghrocca*, which in turn goes around the Earth.

While discussing the geometrical picture of planetary motion, *Āryabhaṭīya-bhāṣya*, as well as *Golasāra* and *Siddhānta-darpaṇa*, consider the orbit of each of the planets individually and they are not put together in a single cosmological model of the planetary system. There is however an interesting passage in *Āryabhaṭīya-bhāṣya*, where Nīlakaṇṭha explains that the Earth is not circumscribed by the orbit of the interior planets, Mercury and Venus; and that the mean period of motion in longitude of these planets around the Earth is the same as that of the Sun, precisely because they are being carried around the Earth by the Sun. In fact, Nīlakaṇṭha seems to be the first savant in the history of astronomy to clearly deduce from his computational scheme—and not from any speculative or cosmological argument—that the interior planets go around the Sun and that the period of their motion around the Sun is also the period of their latitudinal motion.

In a remarkable short tract called *Grahasphuṭānayane vikṣepavāsanā*, which seems to have been written after *Āryabhaṭīya-bhāṣya* as it cites extensively from it, Nīlakaṇṭha succinctly describes his cosmological model, which is that the five planets, Mercury, Venus, Mars, Jupiter and Saturn, go around the mean Sun in eccentric orbits (inclined to the ecliptic), while the mean Sun itself goes around the Earth.[7] Following this, Nīlakaṇṭha also states that the dimensions of *śīghra* epicycles are specified by measuring the orbit of the mean Sun around the Earth in terms of the planetary orbit in the case of the exterior planets, and they are specified by measuring the planetary orbit (which is smaller) in terms of the orbit of the mean Sun in the case of the interior planets. This remarkable relation[8] follows clearly from the identification of the *śighrocca* of all the planets with physical mean Sun, a fact also stated by Nīlakaṇṭha in his *Āryabhaṭīya-bhāṣya*.

Towards the very end of the last chapter of *Tantrasaṅgraha*, Nīlakaṇṭha briefly considers the issue of planetary distances. Unlike the longitudes and latitudes of planets, the planetary distances were not amenable to observations in ancient astronomy and their discussion was invariably based upon some speculative hypothesis. In traditional Indian planetary theory, at least from the time of Āryabhaṭa, the mean planetary distances were obtained based on the hypothesis that all the planets go around the Earth with the same linear velocity—i.e. they all cover the same physical distance in any given period of time. In *Tantrasaṅgraha*, Nīlakaṇṭha proposes an alternative prescription for planetary distances which seems to be based on the principle that all the planets go around the *śighrocca* with the same linear velocity. He also briefly hints at this alternative hypothesis in his *Āryabhaṭīya-bhāṣya*. However, among the available works of Nīlakaṇṭha, there is no discussion of plan-

---

[7] This cosmological model is the same as the one proposed by Tycho Brahe, albeit on entirely different considerations, towards the end of sixteenth century.

[8] The *śīghra* epicycle is essentially the same as the epicycle associated with the so-called 'solar anomaly' in the Greco-European tradition of astronomy, and the above relation is the same as the one proposed by Nicholas Copernicus (perhaps around the same time as Nīlakaṇṭha) by identifying this epicycle as the orbit of the Earth around the Sun in the case of the exterior planets and as the orbit of the planet itself in the case of the interior planets.

etary distances as would follow from his revised cosmological model outlined in *Grahasphuṭānayane vikṣepavāsanā*.

Before taking up the various aspects of the revised planetary model of Nīlakaṇṭha it is essential to understand the traditional Indian planetary model, which had been in vogue at least from the time of Āryabhaṭa (c. 499). We shall therefore devote the initial sections of this appendix to a detailed exposition of the traditional Indian planetary theory and important developments in it prior to the work of Nīlakaṇṭha.

F.1 The traditional Indian planetary model: *Manda-saṃskāra*

In the Indian astronomical tradition, at least from the time of Āryabhaṭa (499 CE), the procedure for calculating the geocentric longitudes of the planets consists essentially of two steps:[9] first, the computation of the mean longitude of the planet known as the *madhyama-graha*, and second, the computation of the true or observed longitude of the planet known as the *sphuṭa-graha*.

The mean longitude is calculated for the desired day by computing the number of mean civil days elapsed since the epoch (this number is called the *ahargaṇa*) and multiplying it by the mean daily motion of the planet. Having obtained the mean longitude, a correction known as *manda-saṃskāra* is applied to it. In essence, this correction takes care of the eccentricity of the planetary orbit around the Sun. The equivalent of this correction is termed the 'equation of centre' in modern astronomy, and is a consequence of the elliptical nature of the orbit. The longitude of the planet obtained by applying the *manda*-correction is known as the *manda-sphuṭa-graha* or simply the *manda-sphuṭa*.

While *manda-saṃskāra* is the only correction that needs to be applied in case of the Sun and the Moon for obtaining their true longitudes (*sphuṭa-grahas*), in the case of the other five planets, two corrections, namely the *manda-saṃskāra* and *śīghra-saṃskāra*, are to be applied to the mean longitude in order to obtain their true longitudes. Here again, we divide the five planets into two groups: the interior, namely Mercury and Venus, and the exterior, namely Mars, Jupiter and Saturn—not necessarily for the purpose of convenience in discussion but also because they are treated differently while applying these corrections.

The *śīghra-saṃskāra* is applied to the *manda-sphuṭa-graha* to obtain the true geocentric longitude known as the *sphuṭa-graha*. As will be seen later, the *śīghra* correction essentially converts the heliocentric longitude into the geocentric longitude. We will now briefly discuss the details of the *manda-saṃskāra*, which will

[9] For a general review of Indian astronomy, see D. A. Somayaji, *A Critical Study of Ancient Hindu Astronomy*, Dharwar 1972; S. N. Sen and K. S. Shukla (eds), *A History of Indian Astronomy*, New Delhi 1985 (rev. edn 2000); B. V. Subbarayappa and K. V. Sarma (eds.), *Indian Astronomy: A Source Book*, Bombay 1985; S. Balachandra Rao, *Indian Astronomy: An Introduction*, Hyderabad 2000; B. V. Subbarayappa, *The Tradition of Astronomy in India: Jyotiḥśāstra*, PHISPC vol. IV, Part 4, Centre for Studies in Civilizations, New Delhi 2008.

be followed by a discussion on the *śīghra-saṃskāra* for the exterior and the interior planets respectively.

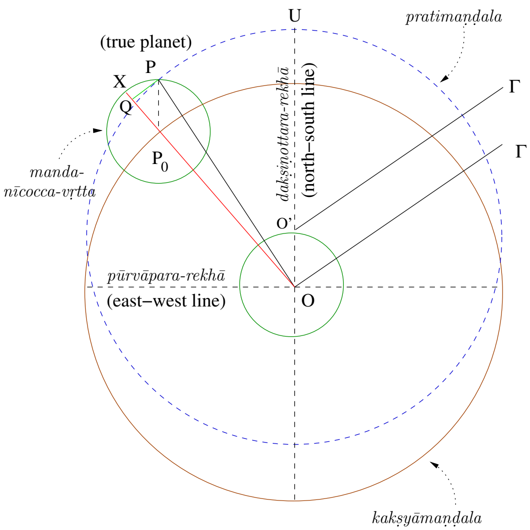

Fig. F.1 The epicyclic and eccentric models of planetary motion.

F.1.1 Epicyclic and eccentric models

As mentioned earlier, the *manda-saṃskāra* essentially accounts for the eccentricity of the planetary orbit. This may be explained with the help of Fig. F.1. Here, O is the centre of the *kakṣyāmaṇḍala*[10] on which the mean planet P_0 is assumed to be moving with mean uniform velocity. $O\Gamma$ is the reference line usually chosen to be the direction of *Meṣādi*. The *kakṣyā-maṇḍala* is taken to be of radius R, known as the *trijyā*.[11] The longitude of the mean planet P_0 moving on this circle is given by

$$\Gamma \hat{O} P_0 = madhyama\text{-}graha = \theta_0. \tag{F.1}$$

The longitude of the *manda-sphuṭa-graha* P given by $\Gamma\hat{O}P$ is to be obtained from θ_0, and this can be obtained by either by an eccentric or epicyclic model.

[10] The centre of the *kakṣyāmaṇḍala* is generally referred to as the *bhagola-madhya* (centre of the celestial sphere), and it coincides with the centre of the Earth in the case of the Sun and the Moon, when the 'second correction' which corresponds to the 'evection term' is ignored.

[11] The value of the *trijyā* is chosen such that one minute of arc in the circle corresponds to unit length. This implies that $2\pi R = 21600$ or $R \approx 3437.74$, which is taken to be 3438 in most of the Indian texts.

The procedure for obtaining the longitude of the *manda-sphuṭa-graha* by either of the two models involves the longitude of the *mandocca*. In Fig. F.1, OU represents the direction of the *mandocca* whose longitude is given by

$$\Gamma \hat{O} U = mandocca = \theta_m. \tag{F.2}$$

The modern equivalent of *mandocca* is *apoapsis*—apogee in the case of the Sun and the Moon and aphelion in the case of the five planets.

Around the mean planet P_0, a circle of radius r is to be drawn. This circle is known as the *manda-nīcocca-vṛtta*[12] or simply as *manda-vṛtta* (epicycle). The texts specify the value of the radius of this circle r $(r \ll R)$, in appropriate measure, for each planet.

At any given instant of time, the *manda-sphuṭa-graha* P is to be located on this *manda-nīcocca-vṛtta* by drawing a line from P_0 along the direction of *mandocca* (parallel to OU). The point of intersection of this line with the *manda-nīcocca-vṛtta* gives the location of the planet P. Since this method of locating the *manda-sphuṭa-graha* involves the construction of an epicycle around the mean planet, it is known as the epicyclic model.

Alternatively, one could draw the *manda-nīcocca-vṛtta* of radius r centred around O, which intersects OU at O'. With O' as centre, a circle of radius R (shown by dashed lines in the figure) is drawn. This is known as *pratimaṇḍala* or the eccentric circle. Since P_0P and OO' are equal to r, and they are parallel to each other, $O'P = OP_0 = R$. Hence, P lies on the eccentric circle. Also,

$$\Gamma \hat{O'} P = \Gamma \hat{O} P_0 = madhyama\text{-}graha = \theta_0. \tag{F.3}$$

Thus, the *manda-sphuṭa-graha* P can be located on an eccentric circle of radius R centred at O' (which is located at a distance r from O in the direction of *mandocca*), simply by marking a point P on it such that $\Gamma \hat{O'} P$ corresponds to the the mean longitude of the planet. Since this process involves only an eccentric circle, without making a reference to the epicycle, it is known as the eccentric model. Clearly, the two models are equivalent to each other.

F.1.2 Calculation of manda-sphuṭa

The formula presented by the Indian astronomical texts for the calculation of the *manda-sphuṭa*—the longitude of the planet obtained by applying the *manda-saṃskāra* (equation of centre) to the mean longitude of the planet—and the underlying geometrical picture can be understood with the help of Fig. F.2.[13] Here,

[12] The adjective *nīcocca* is given to this *vṛtta* because, in this conception, it moves from *ucca* to *nīca* on the deferent circle along with the mean planet P_0. The other adjective *manda* is to suggest that this circle plays a crucial role in the explanation of the *manda-saṃskāra*.

[13] It may be noted that Fig. F.2 is the same as Fig. F.1, with certain circles and markings removed from the latter and certain others introduced in the former for the purposes of clarity.

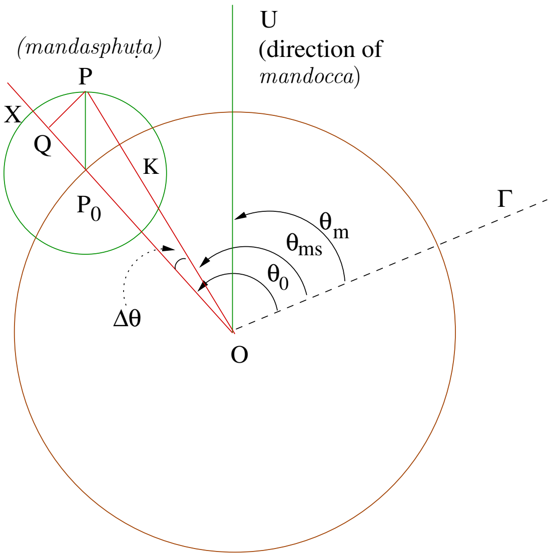

Fig. F.2 Geometrical construction underlying the rule for obtaining the *manda-sphuṭa* from the *madhyama* using the epicycle approach.

$\theta_{ms} = \Gamma\hat{O}P$ represents the *manda-sphuṭa* which is to be determined from the position of the mean planet (*madhyama-graha*) P_0. Clearly,

$$\begin{aligned} \theta_{ms} &= \Gamma\hat{O}P \\ &= \Gamma\hat{O}P_0 - P\hat{O}P_0 \\ &= \theta_0 - \Delta\theta. \end{aligned} \tag{F.4}$$

Since the mean longitude of the planet θ_0 is known, the *manda-sphuṭa* θ_{ms} is obtained by simply subtracting $\Delta\theta$ from the *madhyama*. The expression for $\Delta\theta$ can be obtained by making the following geometrical construction. We extend the line OP_0, which is the line joining the centre of the *kakṣyāmaṇḍala* and the mean planet, to meet the epicycle at X. From P drop the perpendicular PQ onto OX. Then

$$\begin{aligned} U\hat{O}P_0 &= \Gamma\hat{O}P_0 - \Gamma\hat{O}U \\ &= \theta_0 - \theta_m \end{aligned} \tag{F.5}$$

is the *manda-kendra* (*madhyama* – *mandocca*), whose magnitude determines the magnitude of $\Delta\theta$ (see (F.8)). Also, since P_0P is parallel to OU (by construction), $P\hat{P}_0Q = (\theta_0 - \theta_m)$. Hence, $PQ = r\sin(\theta_0 - \theta_m)$ and $P_0Q = r\cos(\theta_0 - \theta_m)$. Since the triangle OPQ is right-angled at Q, the hypotenuse $OP = K$ (known as the *manda-karṇa*) is given by

$$K = OP = \sqrt{OQ^2 + QP^2}$$

$$= \sqrt{(OP_0 + P_0Q)^2 + QP^2}$$
$$= \sqrt{\{R + r\cos(\theta_0 - \theta_m)\}^2 + r^2 \sin^2(\theta_0 - \theta_m)}. \qquad \text{(F.6)}$$

Again from the triangle POQ, we have

$$K \sin \Delta\theta = PQ$$
$$= r \sin(\theta_0 - \theta_m). \qquad \text{(F.7)}$$

Multiplying the above by R and dividing by K we have

$$R \sin \Delta\theta = \frac{r}{K} R \sin(\theta_0 - \theta_m). \qquad \text{(F.8)}$$

In the Āryabhaṭan school, the radius of the *manda* epicycle is assumed to vary in the same way as the *karṇa*, as explained for instance by Bhāskara I (c. 629) in his *Āryabhaṭīya-bhāṣya*, and also in his *Mahābhāskarīya*. Thus the relation (F.8) reduces to

$$R \sin \Delta\theta = \frac{r_0}{R} R \sin(\theta_0 - \theta_m), \qquad \text{(F.9)}$$

where r_0 is the mean or tabulated value of the radius of the *manda* epicycle.

F.1.3 *Aviśiṣṭa-manda-karṇa: iterated hypotenuse*

According to the geometrical picture of planetary motion given by Bhāskara I, the radius of the epicycle *manda-nīcocca-vṛtta* (r) employed in the the *manda* process is not a constant. It varies continuously in consonance with the hypotenuse, the *manda-karṇa* (K), in such a way that their ratio is always maintained constant and is equal to the ratio of the mean epicycle radius (r_0)—whose value is specified in the texts—to the radius of the deferent circle (R). Thus, according to Bhāskara, as far as the *manda* process is concerned, the motion of the planet on the epicycle is such that the following equation is always satisfied:

$$\frac{r}{K} = \frac{r_0}{R}. \qquad \text{(F.10)}$$

If this is the case, then the question arises as to how one can obtain the *manda-karṇa* as well as the the radius of the *manda-nīcocca-vṛtta* at any given instant. For this, Bhāskara provides an iterative procedure called *asakṛt-karma*, by which both r and K are simultaneously obtained. We explain this with the help of Fig. F.3*a*. Here P_0 represents the mean planet around which an epicycle of radius r_0 is drawn. The point P_1 on the epicycle is chosen such that PP_1 is parallel to the direction of the *mandocca*, OU.

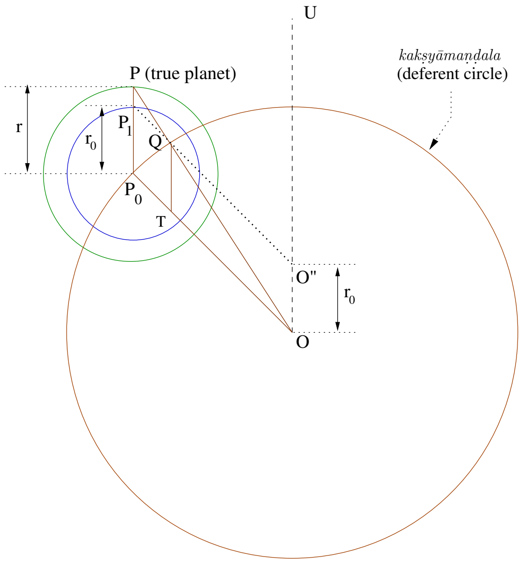

Fig. F.3*a* The variation of the radius of the *manda* epicycle with the *manda-karṇa*.

Now, the first hypotenuse (*sakṛt-karṇa*) is found from r_0 using the relation

$$OP_1 = K_1 = [(R\sin(\theta_0 - \theta_m))^2 + (R\cos(\theta_0 - \theta_m) + r_0)^2]^{\frac{1}{2}}. \tag{F.11}$$

From K_1, using (F.10), we get the next approximation to the radius $r_1 = \frac{r_0}{R}K_1$, and the process is repeated. From r_1 we get the next approximation to the *karṇa*,

$$K_2 = [\{R\sin(\theta_0 - \theta_m)\}^2 + \{R\cos(\theta_0 - \theta_m) + r_1\}^2]^{\frac{1}{2}}, \tag{F.12}$$

and from that we get $r_2 = \frac{r_0}{R}K_2$ and so on, till the radii and the *karṇas* do not change (*aviśeṣa*). The term *aviśeṣa* means 'not distinct'. In the present context it means that the successive *karṇas* are not distinct from each other. That is, $K_{i+1} \approx K_i = K$. If this is satisfied, then $r_{i+1} \approx r_i = r$. Consequently, the equation giving the *manda*-correction (F.8) becomes

$$R\sin\Delta\theta = \frac{r}{K}R\sin(\theta_0 - \theta_m) = \frac{r_0}{R}R\sin(\theta_0 - \theta_m). \tag{F.13}$$

Thus the computation of the *manda-phala* involves only the mean epicycle radius and the value of the *trijyā*. It does not involve the value of the *manda-karṇa*. It can be shown that the iterated *manda-karṇa* is actually given (in the limit) by OP in Fig. F.3*a*, where the point P is obtained as follows.[14] Consider a point O'' at a distance of r_0 from O along the direction of *mandocca* OU and draw $O''P_1$ so that it meets the concentric at Q. Then produce OQ to meet the extension of P_0P_1 at P.

[14] See for instance, the discussion in {MB 1960}, pp. 111–9.

Mādhava of Saṅgamagrāma, the renowned mathematician and astronomer of the 14th century, by carefully analysing the geometry of the problem, came up with a brilliant method of finding the *aviśiṣṭa-manda-karṇa* without performing an iterative process, which is explained in the next section.

F.1.4 Mādhava's formula for the aviśiṣṭa-manda-karṇa

Mādhava's procedure for determining the *aviśiṣṭa-manda-karṇa* involves finding a new quantity called the *viparyaya-karṇa* or *viparīta-karṇa*. The term *viparīta-karṇa* literally means 'inverse hypotenuse', and is nothing but the radius of the *kakṣyāvṛtta* when the *manda-karṇa* is taken to be the *trijyā*, *R*. The following verses from *Tantrasaṅgraha* (II, 43–44) present the way of obtaining the *aviśiṣṭa-manda-karṇa* proposed by Mādhava that circumvents the iterative process.

ा ा तात ो ता ायातप ोा ा ा ायत ा।
ा ा ा ाते ा ा ा ापयय तो ाेत ा ॥
ते ा ता ा ा या ात य ाा ाा तोऽा ा ेष ा यात ।

The square of the *doḥphala* is subtracted from the square of the *trijyā* and its square root is taken. The *koṭiphala* is added to or subtracted from this depending upon whether the *kendra* (anomaly) is within six signs beginning from *Karki* (Cancer) or *Mṛga* (Capricorn). This gives the *viparyaya-karṇa*. The square of the *trijyā* divided by this *viparyaya-karṇa* is the *aviśeṣa-karṇa* (iterated hypotenuse) obtained without any effort [of iteration].

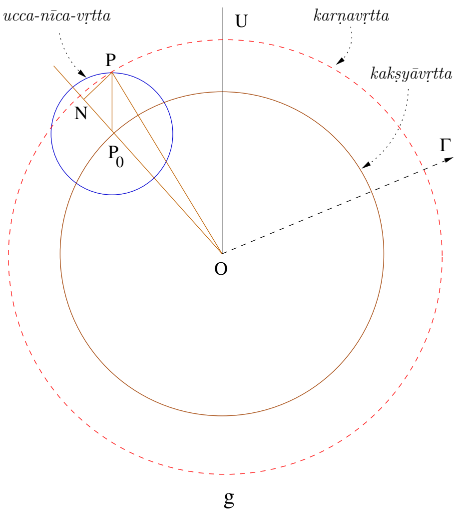

Fig. F.3*b* Determination of the *viparīta-karṇa* when the *kendra* is in the first quadrant.

The rationale behind the formula given for the *viparīta-karṇa* is outlined in the Malayalam text *Yuktibhāṣā*, and can be understood with the help of Figs F.3*a* and F.3*b*. In these figures P_0 and P represent the mean and the true planet respectively. N denotes the foot of a perpendicular drawn from the true planet P to the line joining the centre of the circle and the mean planet. NP is equal to the *doḥphala*. Let the radius of the *karṇavṛtta* OP be set equal to the *trijyā* R. Then the radius of the *uccanīca-vṛtta* P_0P is r_0, as it is in the measure of the *karṇavṛtta*. In this measure, the radius of the *kakṣyāvṛtta* $OP_0 = R_v$, the *viparīta-karṇa*, and is given by

$$\begin{aligned} R_v &= ON \pm P_0N \\ &= \sqrt{R^2 - (r_0 \sin(\theta_0 - \theta_m))^2} \pm |r_0 \cos(\theta_0 - \theta_m)|. \end{aligned} \tag{F.14a}$$

Nīlakaṇṭha has also given another alternative expression for the *viparīta-karṇa* in terms of the longitude θ_{ms} of the *manda-sphuṭa*.

$$R_v = \sqrt{R^2 + r_0^2 - 2Rr_0 \cos(\theta_{ms} - \theta_m)}. \tag{F.14b}$$

This is clear from the triangle OP_0P, where $OP_0 = R_v$, $OP = R$ and $P_0PO = \theta_{ms} - \theta_m$.

In Fig. F.3*a*, Q is a point where $O''P_1$ meets the concentric. OQ is produced to meet the extension of P_0P_1 at P. Let T be the point on OP_0 such that QT is parallel to P_0P_1. Then it can be shown that $OT = R_v$ is the *viparīta-karṇa*. Now, in triangle OQT, $OQ = R$, $QT = P_1P_0 = r_0$ and $O\hat{Q}T = P\hat{O}U = (\theta_{ms} - \theta_m)$ and we have

$$OT = \sqrt{R^2 + r_0^2 - 2Rr_0 \cos(\theta_{ms} - \theta_m)} = R_v. \tag{F.14c}$$

Now, since triangles OQT and OPP_0 are similar, we have

$$\begin{aligned} \frac{OP}{OP_0} &= \frac{OQ}{OT} = \frac{R}{R_v} \\ \text{or,} \quad OP = K &= \frac{R^2}{R_v}. \end{aligned} \tag{F.15}$$

Thus we have obtained an expression for the *aviśiṣṭa-manda-karṇa* in terms of the *trijyā* and the *viparīta-karṇa*. As the computation of the *viparīta-karṇa* as given by (F.14*a*) does not involve iteration, the *aviśiṣṭa-manda-karṇa* can be obtained in one stroke using (F.15) without having to go through the arduous iterative process.

F.1.5 *Manda-saṃskāra* for the exterior planets

We will now discuss the details of the *manda* correction for the case of the exterior planets, namely Mars, Jupiter and Saturn, as outlined in the traditional texts of Indian astronomy. The texts usually specify the the number of revolutions (*bhagaṇas*) made by the planets in a large period known as *Mahāyuga*. In Table F.1, we list the *bhagaṇas* as specified in the texts *Āryabhaṭīya* and *Tantrasaṅgraha*. In the same table, we have also given the corresponding sidereal period of the planet in civil days along with the modern values for the same.

Planet	Revolutions (in *Āryabhaṭīya*)	Sidereal period (in *Āryabhaṭīya*)	Revolutions (in *Tantrasaṅgraha*)	Sidereal period (in *Tantrasaṅgraha*)	Modern values of sidereal period
Sun	4320000	365.25868	4320000	365.25868	365.25636
Moon	57753336	27.32167	57753320	27.32168	27.32166
Moon's apogee	488219	3231.98708	488122	3232.62934	3232.37543
Moon's node	232226	6794.74951	232300	6792.58502	6793.39108
Mercury's *śīghrocca*	17937020	87.96988	17937048	87.96974	87.96930
Venus's *śīghrocca*	7022288	224.69814	7022268	224.70198	224.70080
Mars	2296824	686.99974	2296864	686.98778	686.97970
Jupiter	364224	4332.27217	364180	4332.79559	4332.58870
Saturn	146564	10766.06465	146612	10762.53990	10759.20100

Table F.1 The *bhagaṇas* and sidereal periods of the planets.

In the case of exterior planets, while the planets move around the Sun they also move around the Earth, and consequently, the mean heliocentric sidereal period of the planet is the same as the mean geocentric sidereal period. Therefore, the *madhyama-graha* or the mean longitude of the planet, as obtained from the above *bhagaṇas*, would be the same as the mean heliocentric longitude of the planet as understood today. Now the *manda-saṃskāra* is applied to the *madhyama-graha* to obtain the *manda-sphuṭa-graha*. As we will see below, this *manda* correction is essentially the same as the equation of centre in modern astronomy and thus the *manda-sphuṭa-graha* would essentially be the true heliocentric longitude of the planet.

It was shown above in (F.9) that the magnitude of the correction $\Delta\theta$ to be applied to the mean longitude is given by

$$R\sin\Delta\theta = \frac{r_0}{R} R\sin(\theta_0 - \theta_m), \tag{F.16}$$

If $\frac{r_0}{R}$ is small in the above expression, then $\sin\Delta\theta \ll 1$ and we can approximate $\sin\Delta\theta \approx \Delta\theta$. Hence (F.16) reduces to

$$\Delta\theta = \frac{r_0}{R}\sin(\theta_0 - \theta_m). \tag{F.17}$$

As $\Delta\theta = \theta_0 - \theta_{ms}$, in this approximation we have

$$\theta_{ms} \approx \theta_0 - \frac{r_0}{R}\sin(\theta_0 - \theta_m). \tag{F.18}$$

As outlined in Section F.8.1, in the Keplerean picture of planetary motion the equation of centre to be applied to the mean heliocentric longitude of the planet is given—to the first order in eccentricity—by the equation

$$\Delta\theta \approx (2e)\sin(\theta_0 - \theta_m). \tag{F.19}$$

Now, comparing (F.19) and (F.17), we see that the *manda* correction closely approximates the equation of centre as understood in modern astronomy if the values of $\frac{r_0}{R}$ are fairly close to $2e$.

The values of $\frac{r_0}{R}$ for different planets as specified in *Āryabhaṭīya* and *Tantrasaṅgraha* are listed in Table F.2. It may be noted here that the ratios specified in the texts are close to twice the value of the eccentricity ($2e$) associated with the planetary orbits. In Table F.2, the modern values of $2e$ are listed according to Smart.[15]

Name of the planet	*Āryabhaṭīya* $\frac{r_0}{R}$	Average	*Tantrasaṅgraha* $\frac{r_0}{R}$	Average	$2e$ Modern
Sun	$\frac{13.5}{360}$	0.0375	$\frac{3}{80}$	0.0375	0.034
Moon	$\frac{31.5}{360}$	0.0875	$\frac{7}{80}$	0.0875	0.110
Mercury	$\frac{31.5-9\lvert\sin(\theta_0-\theta_m)\rvert}{360}$	0.075	$\frac{1}{6}$	0.167	0.412
Venus	$\frac{18-9\lvert\sin(\theta_0-\theta_m)\rvert}{360}$	0.0375	$\frac{1}{14+\frac{R\lvert\sin(\theta_0-\theta_m)\rvert}{240}}$	0.053	0.014
Mars	$\frac{63+18\lvert\sin(\theta_0-\theta_m)\rvert}{360}$	0.200	$\frac{7+\lvert\sin(\theta_0-\theta_m)\rvert}{39}$	0.192	0.186
Jupiter	$\frac{31.5+4.5\lvert\sin(\theta_0-\theta_m)\rvert}{360}$	0.0938	$\frac{7+\lvert\sin(\theta_0-\theta_m)\rvert}{82}$	0.091	0.096
Saturn	$\frac{40.5+18\lvert\sin(\theta_0-\theta_m)\rvert}{360}$	0.1375	$\frac{39}{360}$	0.122	0.112

Table F.2 Comparison of *manda* epicycle radii and modern eccentricity values.

F.1.6 *Manda-saṃskāra* for interior planets

For the interior planets Mercury and Venus, since the mean geocentric sidereal period of the planet is the same as that of the Sun, the ancient Indian astronomers took the mean Sun as the *madhyama-graha* or the mean planet. Having taken the mean Sun as the mean planet, they also prescribed the application of the *manda* correction, or the equation of centre characteristic of the planet, to the mean Sun, instead of the mean heliocentric planet. Therefore, the *manda-sphuṭa-graha* in the case of

[15] W. M. Smart, *Textbook on Spherical Astronomy*, Cambridge University Press, 1965, pp. 422–3.

an interior planet, as computed from (F.17) in the traditional planetary model, is just the mean Sun, with a correction applied, and does not correspond to the true heliocentric planet.

However, the ancient Indian astronomers also introduced the notion of the *śīghrocca* for these planets whose period (see Table F.1) is the same as the mean heliocentric sidereal period of these planets. Thus, in the case of the interior planets, it is the longitude of the *śīghrocca* which will be the same as the mean heliocentric longitude of the planet as understood in the currently accepted model of the solar system. As we shall see below, the traditional planetary model made use of this *śīghrocca*, crucially, in the calculation of both the longitudes and latitudes of the interior planets.

F.2 *Śīghra-saṃskāra*

We will now show that the application of *śīghra-saṃskāra* is equivalent to the transformation of the *manda-sphuṭa* to the true geocentric longitude of the planet called the *sphuṭa-graha*. Just as the *mandocca* plays a major role in the application of *manda-saṃskāra*, so too the *śīghrocca* plays a key role in the application of *śīghra-saṃskāra*. As in the case of *manda-saṃskāra*, we shall consider the application of *śīghra-saṃskāra* for the exterior and interior planets separately.

F.2.1 Exterior planets

For the exterior planets, Mars, Jupiter and Saturn, we have already explained that the *manda-sphuṭa-graha* is the true heliocentric longitude of the planet. The *śīghra-saṃskāra* for them can be explained with reference to Fig. F.4*a*. Here A denotes the *nirayaṇa-meṣādi*, E the Earth and P the planet. The mean Sun S is referred to as the *śīghrocca* for exterior planets and thus we have

$$A\hat{S}P = \theta_{ms} \qquad (\textit{manda-sphuṭa})$$
$$A\hat{E}S = \theta_s \qquad (\text{longitude of } \textit{śīghrocca} \text{ (mean Sun)})$$
$$A\hat{E}P = \theta \qquad (\text{geocentric longitude of the planet}).$$

The difference between the longitudes of the *śīghrocca* and the *manda-sphuṭa*, namely

$$\sigma = \theta_s - \theta_{ms}, \tag{F.20}$$

is called the *śīghra-kendra* (anomaly of conjunction) in Indian astronomy. From the triangle EPS we can easily obtain the result

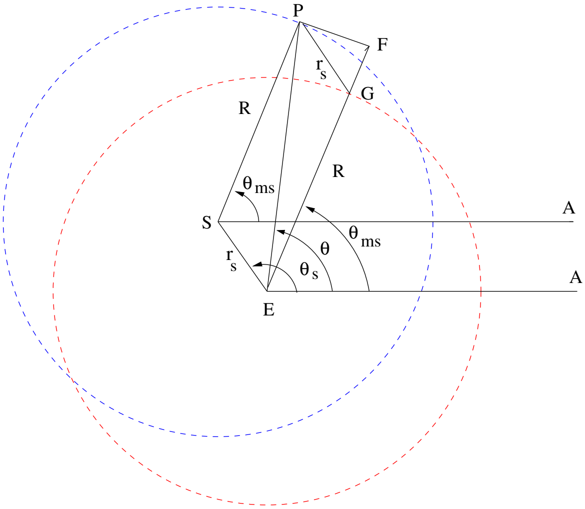

Fig. F.4*a* *Śīghra* correction for exterior planets.

$$\sin(\theta - \theta_{ms}) = \frac{r_s \sin\sigma}{[(R + r_s\cos\sigma)^2 + r_s^2 \sin^2\sigma]^{\frac{1}{2}}}, \qquad \text{(F.21}a\text{)}$$

which is the *śīghra* correction formula given by Indian astronomers to calculate the geocentric longitude of an exterior planet. It may be noted that the true or geocentric longitude of the planet known as the *śīghra-sphuṭa* is found in the same manner from the *manda-sphuṭa*, as the *manda-sphuṭa* is found from the mean planet, the *madhyama-graha*.

From Fig. F.4*a* it is clear that the *śīghra-saṃskāra* transforms the true heliocentric longitudes into geocentric longitudes only if the ratio of the radii of the epicycle and the deferent circle is equal to the ratio of the Earth–Sun and planet–Sun distances. That this is indeed very nearly so in the Indian texts, as may be seen from Table F.3. It my also be noted that (F.21*a*) has the same form as the formula for the difference between the geocentric and heliocentric longitudes for an exterior planet in the Keplerian model (see (F.46)) if $\frac{r_s}{R}$ is identified with the ratio of the Earth–Sun and planet–Sun distances. However, (F.21*a*) is still an approximation as it is based upon mean Sun and not the true Sun.

F.2.2 Interior planets

The *śīghra-saṃskāra* for the interior planets can be explained with reference to Fig. F.4*b*. Here E is the Earth and S (the *manda*-corrected mean Sun) is the *manda-sphuṭa-graha* and P, the so-called *śīghrocca*, actually corresponds to the (mean heliocentric) planet. We have

$$A\hat{E}S = \theta_{ms} \qquad (\textit{manda-sphuṭa})$$
$$A\hat{S}P = \theta_s \qquad (\text{longitude of } \textit{śīghrocca})$$
$$A\hat{E}P = \theta \qquad (\text{geocentric longitude of the planet}).$$

Again, the *śīghra-kendra* is defined as the difference between the *śīghrocca* and the *manda-sphuṭa-graha* as in (F.20). Thus, from the triangle *EPS* we get the same formula

$$\sin(\theta - \theta_{ms}) = \frac{r_s \sin\sigma}{[(R + r_s\cos\sigma)^2 + r_s^2\sin^2\sigma]^{\frac{1}{2}}}, \qquad \text{(F.21}b\text{)}$$

which is the *śīghra* correction given in the earlier Indian texts to calculate the geocentric longitude of an interior planet. For the interior planets also, the value specified for $\frac{r_s}{R}$ is very nearly equal to the ratio of the planet–Sun and Earth–Sun distances, as may be seen from Table F.3.

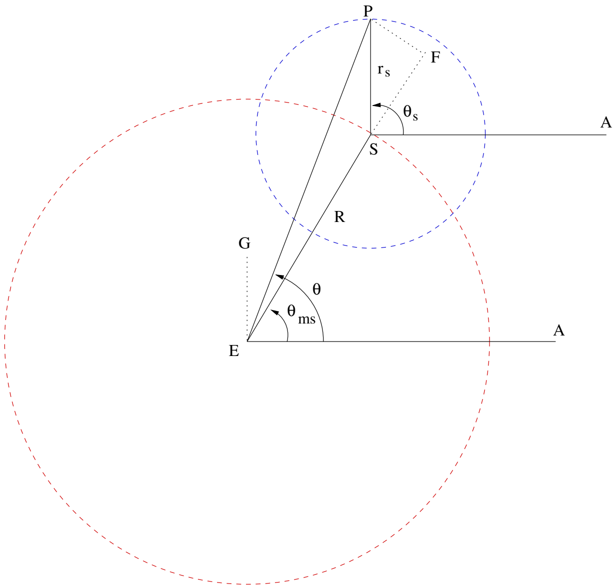

Fig. F.4*b* *Śīghra* correction for interior planets.

Since the *manda* correction or equation of centre for an interior planet was applied to the longitude of the mean Sun instead of the mean heliocentric longitude of the planet, the accuracy of the computed longitudes of the interior planets according to the ancient Indian planetary models would not have been as good as that achieved for the exterior planets. But for the wrong application of the equation of centre, equation (F.21*b*) has the same form as the formula for the difference between the geocentric longitude of an interior planet and the Sun in the Keplerian model (see (F.50)), if $\frac{r_s}{R}$ is identified with the ratio of the planet–Sun and Earth–Sun distances.

Name of the planet	*Āryabhaṭīya* $\frac{r_s}{R}$	Average	*Tantrasaṅgraha* $\frac{r_s}{R}$	Average	Modern value
Mercury	$\frac{139.5-9\lvert\sin(\theta_{ms}-\theta_s)\rvert}{360}$	0.375	$\frac{133-\lvert\sin(\theta_{ms}-\theta_s)\rvert}{360}$	0.368	0.387
Venus	$\frac{265.5-9\lvert\sin(\theta_{ms}-\theta_s)\rvert}{360}$	0.725	$\frac{59-2\lvert\sin(\theta_{ms}-\theta_s)\rvert}{80}$	0.725	0.723
Mars	$\frac{238.5-9\lvert\sin(\theta_{ms}-\theta_s)\rvert}{360}$	0.650	$\frac{7+\lvert\sin(\theta_{ms}-\theta_s)\rvert}{39}$	0.656	0.656
Jupiter	$\frac{72-4.5\lvert\sin(\theta_{ms}-\theta_s)\rvert}{360}$	0.194	$\frac{16-\lvert\sin(\theta_{ms}-\theta_s)\rvert}{80}$	0.194	0.192
Saturn	$\frac{40.5-4.5\lvert\sin(\theta_{ms}-\theta_s)\rvert}{80}$	0.106	$\frac{9-\lvert\sin(\theta_{ms}-\theta_s)\rvert}{80}$	0.106	0.105

Table F.3 Comparison of $\frac{r_s}{R}$, as given in *Āryabhaṭīya* and *Tantrasaṅgraha*, with the modern values of the ratio of the mean values of Earth–Sun and planet–Sun distances for the exterior planets and the inverse ratio for the interior planets.

F.2.3 Four-step process

In obtaining the expression (F.21) for the *śīghra* correction, we had taken SP, the Sun–planet distance, to be given by R. But actually SP is a variable and is given by the (iterated) *manda-karṇa* K. Hence the correct form of the *śīghra* correction should be

$$\sin(\theta_s-\theta_{ms}) = \frac{r_s\sin\sigma}{\{(K+r_s\cos\sigma)^2+r_s^2\sin^2\sigma\}^{\frac{1}{2}}}, \tag{F.22}$$

where K is the (iterated) *manda-karṇa*. Since K as given by (F.14) and (F.15) depends on the *manda* anomaly $\theta-\theta_m$, the *śīghra* correction as given by (F.22) cannot be tabulated as a function of the *śīghra* anomaly (σ) alone.

It is explained in *Yuktibhāṣā* (section 8.20) that, in order to simplify computation, the ancient texts on astronomy advocated that the computation of the planetary longitudes may be done using a four-step process—involving half-*manda* and half-*śīghra* corrections followed by the full *manda* and *śīghra* corrections. The *śīghra* corrections involved in the four-step process are based on the simpler formula (F.21) which can be read off from a table. According to *Yuktibhāṣā*, the results of the four-step process indeed approximate those obtained by the application of the *manda* correction followed by the *śīghra* correction where, in the latter correction, the effect of the *manda-karṇa* is properly taken into account as in (F.22).

F.2.4 Computation of planetary latitudes

Planetary latitudes (called *vikṣepa* in Indian astronomy) play an important role in the prediction of planetary conjunctions, the occultation of stars by planets etc. In Fig. F.5, P denotes the planet moving in an orbit inclined at an angle i to the ecliptic, intersecting the ecliptic at point N, the node (called the *pāta* in Indian astronomy). If β is the latitude of the planet, θ_h its heliocentric longitude and θ_n the heliocentric

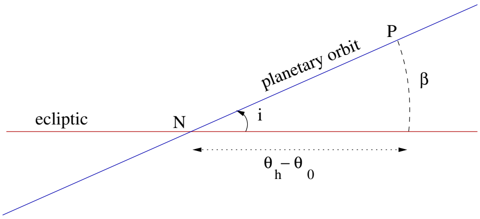

Fig. F.5 Heliocentric latitude of a planet.

longitude of the node, then it can be shown that

$$\sin\beta = \sin i \, \sin(\theta_h - \theta_n). \tag{F.23}$$

For small i we have

$$\beta = i \, \sin(\theta_h - \theta_n). \tag{F.24}$$

This is essentially the rule for calculating the latitude of a planet, as given in Indian texts, at least from the time of Āryabhaṭa.[16] For the exterior planets, it was stipulated that

$$\theta_h = \theta_{ms}, \tag{F.25}$$

the *manda-sphuṭa-graha*, which, as we saw earlier, coincides with the heliocentric longitude of the exterior planet. The same rule applied for interior planets would not have worked, because in the traditional Indian planetary model the *manda*-corrected mean longitude for the interior planet has nothing to do with its true heliocentric longitude. However, most of the Indian texts on astronomy stipulated that the latitude in the case of the interior planets is to be calculated from (F.24) with

$$\theta_h = \theta_s + \textit{manda}\ \text{correction}, \tag{F.26}$$

the *manda*-corrected longitude of the *śīghrocca*. Since the longitude of the *śīghrocca* for an interior planet, as we explained above, is equal to the mean heliocentric longitude of the planet, (F.26) leads to the correct relation that, even for an interior planet, θ_h in (F.24) becomes identical with the true heliocentric longitude. Thus we see that the earlier Indian astronomical texts did provide a fairly accurate theory for the planetary latitudes. But they had to live with two entirely different rules for calculating latitudes: one for the exterior planets given by (F.25), where the *manda-sphuṭa-graha* appears; and an entirely different one for the interior planets given by (F.26), which involves the *śīghrocca* of the planet, with the *manda* correction included.

This peculiarity of the rule for calculating the latitude of an interior planet was noticed repeatedly by various Indian astronomers, at least from the time of

[16] Equation (F.24) actually gives the heliocentric latitude and needs to be multiplied by the ratio of the geocentric and heliocentric distances of the planet to get the geocentric latitude. This feature was implicit in the traditional planetary models.

Bhāskara I (c. 629), who in his *Āryabhaṭīya-bhāṣya* drew attention to the fact that the procedure given in *Āryabhaṭīya* for calculating the latitude of an interior planet is indeed very different from that adopted for the exterior planets.[17] The celebrated astronomer Bhāskarācārya II (c. 1150) also draws attention to this peculiar procedure adopted for the interior planets, in his *Vāsanā-bhāṣya* on his own *Siddhāntaśiromaṇi*, and quotes the statement of Caturveda Pṛthūdakasvāmin (c. 860) that this peculiar procedure for the interior planets can be justified only on the ground that this is what has been found to lead to predictions that are in conformity with observations.[18]

F.3 Geometrical picture of planetary motion according to Parameśvara

The renowned Kerala astronomer Parameśvara of Vaṭasseri (1380–1460) has discussed in detail the geometrical model implied in the conventional planetary model of Indian astronomy in his super-commentary *Siddhāntadīpikā* (on Govindasvāmin's commentary on) *Mahābhāskarīya* of Bhāskara I. A shorter version is available in his commentary on *Āryabhaṭīya*, which is given below.

ाधया त ध्ये ि ि ि ि ि ि ि ि ि।
त िि ि ि पि ि प्र यते तेषा ि॥
ि ि या त ध्य या ि ि ि ि ि त ्य ि।
ि्र ि ि त य ि्र त्य त ते प ि ि॥
ि ि ि ि ि ि्र प्रात ि त य ि ता ि ि।
ि ि ि मा या ि ि प ि त तट त॥
ि ि ि त्य त ते ि ि िप ि ि ि ि।
यात प्रात ि त ि ि त ि ि ि ्र पि ॥
ि प्रात ि ि तट यायात या ि य ते।
ता ि तेषा ि ि प्रा ि त ि ि ि्र ते॥
प्रात ि ि िता य ते य ते प ि त ि्रा ये।
या ि ि या ि ि ि ता ि ि ते य।
ए ि ि ि िित ता ि य ि ता ि िित ि।
या ि ि य ते ता ि त िता यतोऽ या ि ि॥
ि यतेऽ ि ता ि ि ि ध्ये ि ध िप ि्राध ि।
ि्र ि ि ि्र ित ि ितोऽ या ि॥
ि या ि ि प्र ि ि बध ि यो ध्य यात।
तट त म ि ि ि ि ि त्य त ते त ध्य यात।
ि प्रात ि य ता ि ि या ि ि ितो त ि ता ि।
प्रात ि य ध्य ि्र य त य ि ि िप ि ि त ि।

---

[17] {AB 1976}, p. 32, 247.

[18] {SSR 1981}, p. 402.

ाप्र ा ा ित यता ा ा त ा ा ा ा ै ा।
॰ ॅ या प्रा ाि यात ो पृ ा ा ेा ा॥
ा यतेऽ ा ता ा ा ा ा ेप्राधृ यत्यये ा ा ृेचे।
ताि ा ा प्रा प ा ॅप्र ा ृा ा ा पृो ॥[19]

Since the rationale for the *sphuṭavidhi* (the scheme of computing the true planet) for the celestial bodies is not clear without the aid of *chedyaka* (diagrams), we present briefly the way of obtaining the diagrams.

For Mars, Jupiter and Saturn, with the centre of the Earth as the centre, the *śīghra-kakṣyā-vṛtta* (concentric circle) is drawn with the *trijyā* ($R\sin 90$) as the radius. Then draw the *śīghra-pratimaṇḍala* (eccentric circle) with its centre located at a distance of the *śīghra-antyaphala* (maximum *śīghra*-correction) in the direction of the *śīghrocca*. The same will be the *manda*-concentric. From its centre go along in the direction of the *mandocca* a distance equal to the maximum *manda* correction, and with this as the centre draw a circle. This is referred as the *manda* eccentric circle. The planets Mars, Jupiter and Saturn move on this eccentric when reduced to the *manda*-concentric they are referred to as *manda-sphuṭa*, and when reduced to the *śīghra*-concentric they are *sphuṭa* (true planets). ...

For Mercury and Venus, the *manda*-concentric is first drawn with the centre of the Earth as the centre. From that go along in the direction of *mandocca* a distance equal to the maximum *manda* correction and with that as the centre draw the *manda* eccentric circle. The point where the Sun is located on that eccentric is the centre of the *śīghra* epicycle and the radius of that circle is [not the *trijyā* but] as enunciated. In that *śīghra* epicycle, the Mercury and the Venus always move ...

The *chedyaka* procedure enunciated by Parameśvara is illustrated in Figs F.6 and F.7. In both these figures, O represents the observer, M the *mandocca* and P the planet whose longitude as measured from O is to be determined. In Fig. F.6, the circles C_1, C_2 and C_3 are all of radius R. The circle C_1, centred around the observer O, is the *śīghra-kakṣyā-maṇḍala* or the *śīghra*-concentric circle. The circle C_2 which is centred at the *śīghrocca* S is the *śīghra-pratimaṇḍala* (*śīghra*-eccentric circle). The distance of separation between these two circles denoted by OS is the *śīghrāntya-phala*, and corresponds to the radius of the *śīghra* epicycle. It has been clearly enunciated by Parameśvara that the *śīghra-pratimaṇḍala*, denoted by C_2 in the figure, itself serves as the *manda-kakṣyā-maṇḍala*, or the *manda*-concentric circle. The third circle C_3, which is centred around the *mandocca* M, is the *manda-pratimaṇḍala* or the *manda*-eccentric circle. The distance of separation between the centers of the *manda*-concentric and the *manda*-eccentric circles is equal to the radius of the *manda* epicycle and is also the *mandāntya-phala*, whose measure varies from planet to planet.

Parameśvara has depicted the geometrical picture of motion of the interior planets also by employing three circles, C_1, C_2 and C_3, as in the case of exterior planets, as shown in Fig. F.7. However, here these three circles have completely different connotations and, while C_1 and C_2 are of radius R, C_3 is of radius r_s, the radius of the *śīghra* epicycle. Here the circle C_1 centred around O, is the *manda-kakṣyā-maṇḍala*, or the *manda*-concentric circle. The circle C_2, which is centred around the *mandocca* M, is the *manda-pratimaṇḍala*, which serves as the locus for the

[19] {AB 1874}, pp. 60-1.

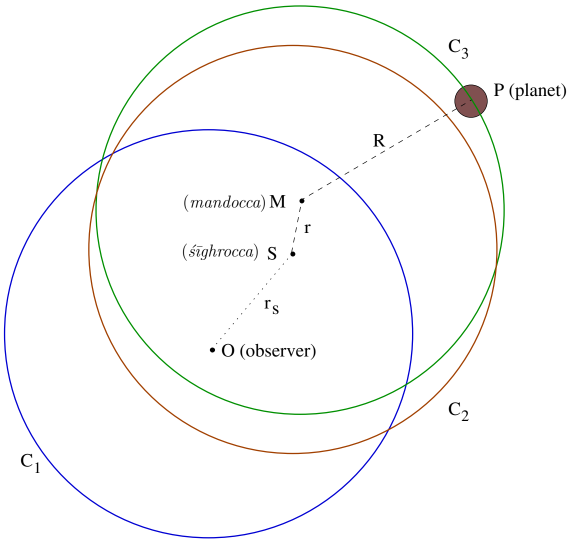

Fig. F.6 Geometrical picture of the motion of an exterior planet given by Parameśvara.

centre of the *śīghra-vṛtta* denoted by the circle C_3. The distance of separation between the centers of C_1 and C_2 is equal to the radius of the *manda* epicycle, and is also the *mandāntya-phala*. P represents the *śīghrocca* associated with the interior planet and S is the *manda*-corrected Sun on the *manda-pratimaṇḍala*.

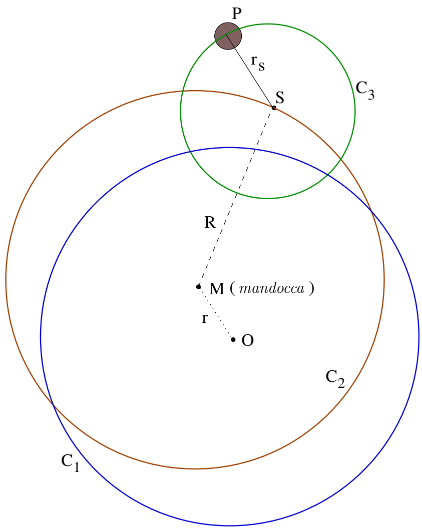

Fig. F.7 Geometrical picture of the motion of an interior planet given by Parameśvara.

It is important to note that, through his diagrammatic procedure, Parameśvara clearly illustrates the fact that, in the traditional planetary model, the final longitude that is calculated for an interior planet is actually the geocentric longitude of what is called the *śīghrocca* of the planet. From Figs F.6 and F.7 we can see easily that Parameśvara's geometrical picture of planetary motion is fairly accurate except for the fact that the equation of centre for the interior planets is wrongly applied to the mean Sun. Incidentally, it may also be noted that Parameśvara has given a succinct description of the same *chedyakavidhi* in his *Goladīpikā*.[20]

F.4 Nīlakaṇṭha's revised planetary model

Among the available works of Nīlakaṇṭha, his revised planetary motion is discussed in the works *Tantrasaṅgraha*, *Āryabhaṭīya-bhāṣya*, *Siddhānta-darpaṇa* and the *Vyākhyā* on it, *Golasāra* and the tract called *Grahasphuṭānayane vikṣepavāsanā*. Of these, *Golasāra* and *Siddhānta-darpaṇa* are presumed to have been written prior to the detailed work *Tantrasaṅgraha* composed in 1500. The *Āryabhaṭīya-bhāṣya* refers to *Golasāra* and *Tantrasaṅgraha*. The *Siddhānta-darpaṇa-vyākhyā* cites the *Āryabhaṭīya-bhāṣya*. In the same way, the small but important tract *Grahasphuṭānayane vikṣepavāsanā* includes long passages from the *Āryabhaṭīya-bhāṣya* and is clearly a later composition.

In *Tantrasaṅgraha*, Nīlakaṇṭha presents the revised planetary model and also gives the detailed scheme of computation of planetary latitudes and longitudes, but he does not discuss the geometrical picture of planetary motion. Towards the end of the last chapter of the work, Nīlakaṇṭha introduces a prescription for the *sphuṭakakṣyā* (the true distance of the planets). There seems to be just a brief (and incomplete) mention of this subject in *Golasāra* and *Āryabhaṭīya-bhāṣya*.

The geometrical picture of planetary motion is discussed in detail in the *Āryabhaṭīya-bhāṣya*. It is also succinctly presented in terms of a few verses in both *Golasāra* and *Siddhānta-darpaṇa*. Nīlakaṇṭha presents some aspects of his cosmological model while discussing the geometrical picture of the motion of the interior planets in his *Āryabhaṭīya-bhāṣya*. He presents a definitive but succinct account of his cosmological model in terms of a few verses in his later work *Grahasphuṭānayane vikṣepavāsanā*.

F.4.1 Identifying the mean Mercury and Venus

In the very first chapter of *Tantrasaṅgraha* (c. 1500), Nīlakaṇṭha introduces a major revision of the traditional Indian planetary model, according to which what were traditionally referred to as the *śīghrocca*s of the interior planets (Mercury and

[20] {GD 1916}, pp. 14–15.

Venus) are now identified with the planets themselves; and the mean Sun is taken as the *śīghrocca* of all the planets.

ाा ि ेष ा ाा ा ा े े, ा य त ।
े ाडा ाङ्का ा ा, ा य ापययया ॥
ाा ाे ा ा ा ा ा ाङ्क ा ा ाय ।
यो ाा रूपो ाडपा ा ा ा ब पते ॥
ाड ाेाा ा ाा यो ा ापययया ।[21]

[The number of revolutions in a *mahāyuga*] of the Moon is 57753320. That of Mars is 2296864. The number of own revolutions of Mercury is 17937048. That of Jupiter is 364180. The number of revolutions of Venus is 7022268.

Here the commentator Śaṅkara Vāriyar observes:

ा ा ाड ापयायाा ाा ाा ा ाायाा ाा ा ात ा ाात्रोच्च ाम्बा धटाबध य ा ा त ा[22]

Here, by the use of the word *sva* (own), the association of this number of revolutions with the *śīghrocca* of Mercury, as done by Bhāskara and others, is rejected.

It may be noted (see Table F.1) that, except for the above redefinition of the mean Mercury and Venus, the *bhagaṇas*, or the number of planetary revolutions in a *Mahāyuga*, are nearly same as those given in *Āryabhaṭīya*.

F.4.2 Computation of planetary longitudes

Nīlakaṇṭha presents the details of his planetary model in the second chapter of *Tantrasaṅgraha*. For the exterior planets, he essentially follows the traditional model. He also retains the four-step process, while noting that (the rationale for such a scheme seems to be essentially that) such has been the recommendation of the earlier masters:

ाा ेत्र प ा ाा ेत्र ाटााय ा ाात ।
ा ा ा ाा ाा ा ाा य ाा ा ाा ा ा ॥[23]

The earlier masters have stated that the *manda*, *śīghra* and again *manda* and *śīghra* are the four corrections that have to be applied in sequence in the case of Mars, Jupiter and Saturn (in order to obtain their geocentric longitude).

The actual procedure given by Nīlakaṇṭha is the following: If θ_0 is the mean longitude of the planet and θ_m that of its *mandocca*, then θ_1 (the longitude at the end of the first step of the four-step process) is found by applying the half-*manda* correction as follows:

---

[21] {TS 1958}, p. 8.
[22] {TS 1958}, p. 9.
[23] {TS 1958}, p. 41.

$$\theta_1 = M + \frac{1}{2} R\sin^{-1}\left[-\frac{r_0}{R} R\sin(\theta_0 - \theta_m)\right], \qquad \text{with}$$

$$\frac{r_0}{R} = \frac{[7 + |\sin(\theta_0 - \theta_m)|]}{39} \qquad \text{(for Mars)}$$

$$\frac{r_0}{R} = \frac{[7 + |\sin(\theta_0 - \theta_m)|]}{82} \qquad \text{(for Jupiter)}$$

$$\frac{r_0}{R} = \frac{39}{320} \qquad \text{(for Saturn).}$$

Then θ_2 is found by applying the half-*śīghra* correction with the mean Sun θ_s as the *śīghrocca* as follows:

$$\theta_2 = \theta_1 + \frac{1}{2} R\sin^{-1}\left[\frac{r_s}{K_{s1}} R\sin(\theta_s - \theta_1)\right], \qquad \text{with}$$

$$K_{s1} = [\{r_s \sin(\theta_1 - \theta_s)\}^2 + \{R + r_s \cos(\theta_1 - \theta_s)\}^2]^{\frac{1}{2}}$$

$$\left(\frac{r_s}{R}\right) = \frac{[53 - 2|\sin(\theta_1 - \theta_s)|]}{80} \qquad \text{(for Mars)}$$

$$\left(\frac{r_s}{R}\right) = \frac{[16 - |\sin(\theta_1 - \theta_s)|]}{80} \qquad \text{(for Jupiter)}$$

$$\left(\frac{r_s}{R}\right) = \frac{[9 - |\sin(\theta_1 - \theta_s)|]}{80} \qquad \text{(for Saturn).}$$

Then the *manda-sphuṭa* θ_{ms} is found by adding the whole *manda* correction obtained with θ_2 to θ_0:

$$R\sin(\theta_{ms} - \theta_0) = -\left(\frac{r_0}{R}\right) R\sin(\theta_2 - \theta_m).$$

Then the true planet *sphuṭa-graha* P is found by applying the whole of the *śīghra* correction to θ_{ms}.

$$R\sin(\theta - \theta_{ms}) = \left[\frac{r_s}{K_s} R\sin(\theta_s - \theta_{ms})\right]$$

$$\text{where} \qquad K_s = [\{r_s \sin(\theta_{ms} - \theta_s)\}^2 + \{R + r_s \cos(\theta_{ms} - \theta_s)\}^2]^{\frac{1}{2}}. \qquad \text{(F.27)}$$

Again, as we had noted earlier in connection with the traditional planetary model, in the above four-step process also the iterated *manda*-hypotenuse (*aviśiṣṭa-manda-karṇa*) does not appear and the *manda* and *śīghra* corrections can be read off from a table.

In the case of the interior planets, Nīlakaṇṭha presents just the two-step process: *manda-saṃskāra* followed by *śīghra-saṃskāra*. For the interior planets, if θ_0 is the longitude of the mean planet (as per his revised model), θ_m its *mandocca* and θ_s that of the mean Sun (*śīghrocca*), then the *manda* correction leading to the *mandasphuṭa* is given by

$$R\sin(\theta_{ms} - \theta_0) = -\frac{r_0}{R} R\sin(\theta_0 - \theta_m)$$

$$\frac{r_0}{R} = \frac{1}{6}, \frac{1}{\left[14 + \frac{|R\sin(\theta_0 - \theta_m)|}{240}\right]} \qquad \text{(for Mercury, Venus).}$$

It may be recalled that the *aviśiṣṭa-manda-karṇa* K is to be calculated using the Mādhava formula (F.15). The *śīghra* correction giving the true planet θ is given by

$$R\sin(\theta - \theta_s) = \left[\left(\frac{r_s}{R}\right)\left(\frac{K}{K_s}\right) R\sin(\theta_{ms} - \theta_s)\right]$$

$$\text{where} \qquad K_s = [R\sin(\theta_{ms} - \theta_s)^2 + \{R\cos(\theta_{ms} - \theta_s) + \left(\frac{r_s}{R}\right) K\}^2]^{\frac{1}{2}} \qquad \text{(F.28)}$$

$$\left(\frac{r_s}{R}\right) = \frac{[31 - 2|\sin(\theta_{ms} - \theta_s)|]}{80R} \qquad \text{(for Mercury)}$$

$$\left(\frac{r_s}{R}\right) = \frac{[59 - 2|\sin(\theta_{ms} - \theta_s)|]}{80R} \qquad \text{(for Venus).}$$

Note that in the above two-step process the *aviśiṣṭa-manda-karṇa* K shows up in the *śīghra* correction. In his discussion of the geometrical picture of planetary motion in the *Āryabhaṭīya-bhāṣya*, Nīlakaṇṭha presents the two-step process as the planetary model for all the planets. This has also been the approach of *Yuktibhāṣā*.

F.4.3 Planetary latitudes

In the seventh chapter of *Tantrasaṅgraha*, Nīlakaṇṭha gives the method for calculating the latitudes of planets, and prescribes that for all planets, both exterior and interior, the latitude is to be computed from the *manda-sphuṭa-graha*.

ा ात ापातो ाात ौ ाा ा ाा ा ाा ा ाात।
प ा ोपा ा्ा यात ोपोऽ त्य ा ो त ॥[24]

> The Rsine of the *manda-sphuṭa* of the planet Mars etc., from which the longitude of its node is subtracted, is multiplied by the maximum latitude and divided by the last hypotenuse (the *śīghra* hypotenuse of the last step). The result is the latitude of the planet.

This is as it should be, for in Nīlakaṇṭha's model the *manda-sphuṭa-graha* (the *manda* corrected mean longitude) coincides with the true heliocentric longitude for both exterior and interior planets. In this way, Nīlakaṇṭha, by his modification of the traditional Indian planetary theory, solved the problem, long-standing in Indian astronomy, of there being two different rules for calculating the planetary latitudes.

In the above verse, Nīlakaṇṭha states that the last hypotenuse that arises in the process of computation of longitudes, namely the *śīghra-karṇa* K_s, is to be used as the divisor. In *Āryabhaṭīya-bhāṣya*, he identifies this as the Earth–planet distance (the *bhū-tārāgraha-vivara*). There, Nīlakaṇṭha has also explained how the computations of true longitude and latitude get modified when latitudinal effects are also

[24] {TS 1958}, p. 139.

taken into account. The true Earth-planet distance (the *bhū-tārāgraha-vivara*) is also calculated there in terms of the K_s and the latitude.[25]

From the above discussion it is clear that the central feature of Nīlakaṇṭha's revision of the traditional planetary model is that the *manda* correction, or the equation of centre for the interior planets, should be applied to the mean heliocentric planet (or what was referred to as the *śīghrocca* in the traditional Indian planetary model), and not the mean Sun. In this way Nīlakaṇṭha, by 1500 CE, had arrived at the correct formulation of the equation of centre for the interior planets, perhaps for the first time in the history of astronomy. Nīlakaṇṭha was also able to formulate a unified theory of planetary latitudes.

Just as was the case with the earlier Indian planetary model, the ancient Greek planetary model of Ptolemy and the planetary models developed in the Islamic tradition during the 8th–15th centuries postulated that the equation of centre for an interior planet should be applied to the mean Sun, rather than to the mean heliocentric longitude of the planet as we understand today.[26] Further, while the ancient Indian astronomers successfully used the notion of the *śīghrocca* to arrive at a satisfactory theory of the latitudes of the interior planets, the Ptolemaic model is totally off the mark when it comes to the question of latitudes of these planets.[27]

Even the celebrated Copernican revolution brought about no improvement in the planetary theory for the interior planets. As is widely known now, the Copernican model was only a reformulation of the Ptolemaic model—with some modifications borrowed from the Maragha school of astronomy of Nasir ad-Din at-Tusi (c. 1201–74), Ibn ash-Shatir (c. 1304–75) and others—for a heliocentric frame of reference, without altering his computational scheme in any substantial way for the interior planets. As an important study notes:

> 'Copernicus, ignorant of his own riches, took it upon himself for the most part to represent Ptolemy, not nature, to which he had nevertheless come the closest of all'. In this famous and just assessment of Copernicus, Kepler was referring to the latitude theory of Book V [of *De Revolutionibus*], specifically to the 'librations' of the inclinations of the planes of the eccentrics, not in accordance with the motion of the planet but by the unrelated motion of the Earth. This improbable connection between the inclinations of the orbital planes and the motion of the Earth was the result of Copernicus's attempt to duplicate the apparent latitudes of Ptolemy's models in which the inclinations of the epicycle planes were variable. In a way this is nothing new since Copernicus was also forced to make the equation of centre of the interior planets depend upon the motion of the Earth rather than the planet.[28]

Indeed, it appears that the correct rule for applying the equation of centre for an interior planet to the mean heliocentric planet (as opposed to the mean Sun), and a

[25] {ABB 1957}, pp. 6–7. This issue has also been discussed at great length in {GYB 2008}, pp. 495–500, 653–9, 883–9).

[26] See for example *The Almagest by Ptolemy*, translated by G. J. Toomer, London 1984.

[27] As a well-known historian of astronomy has remarked: 'In no other part of planetary theory did the fundamental error of the Ptolemaic system cause so much difficulty as in accounting for the latitudes, and these remained the chief stumbling block up to the time of Kepler' (J. L. E. Dreyer, *A History of Astronomy from Thales to Kepler*, New York 1953, p. 200).

[28] N. M. Swerdlow and O. Neugebauer, *Mathematical Astronomy in Copernicus' De Revolutionibus*, Part I, New York 1984, p. 483.

satisfactory theory of latitudes for the interior planets, were first formulated in the Greco-European astronomical tradition only in the early 17th century by Kepler.

We have already seen how the traditional Indian planetary model presented a fairly accurate computational scheme for calculating longitudes and latitudes for the exterior planets. With his revision of the traditional model, Nīlakaṇṭha arrived at a fairly accurate scheme for the interior planets also. In fact, as a computational scheme for calculating planetary longitudes and latitudes, Nīlakaṇṭha's model is indeed a good approximation to the Keplerian model of planetary motion.

F.4.4 Rationale for the revised planetary model

In his *Āryabhaṭīya-bhāṣya*, Nīlakaṇṭha explains the rationale behind his revision of the traditional planetary theory. This has to do with the fact (which, as we have mentioned above, was also noticed by several Indian astronomers prior to Nīlakaṇṭha) that the traditional planetary model employed entirely different schemes for computing the latitudes of the exterior and the interior planets. While the latitude of the exterior planets was computed from their so-called *manda-sphuṭa* (which corresponds to what we currently refer to as the true heliocentric planet), the latitudes of the interior planets was computed from their so-called *śīghrocca*. Nīlakaṇṭha argued that since the latitude should be dependent upon the deflection (from the ecliptic) of the planet itself and not of any other body, what was traditionally referred to as the *śīghrocca* of an interior planet should be identified with the planet itself. Nīlakaṇṭha also showed that this would lead to a unified treatment of the latitudinal motion of all the planets—interior as well as exterior.

In his commentary on verse 3 of *Golapāda* of Āryabhaṭa dealing with the calculation of latitudes, Nīlakaṇṭha discusses the special features that arise in the case of interior planets. It is here that he provides a detailed rationale for his revision of the traditional planetary model:

> ाक्र ाच्च ाोप ˘ । ˘ ोत यते? ा ाबम्ब य ाोप ाक्र ा ा ाॅ ा
> ा ात ा ात, ा प ा य क्र ा ा ाा ात। ात्य ा। ा प ा य य क्र ा ा ात
> य य ा ोप पप ातेात ातबध ात्यै ा ा ोे ाक्र ा ा ाप यात।...
>
> एतच्च ोपप ाते य॑॑ ोे ा ाट ाॅ ा तत्पा ˘क्र ा ा ाप ˘ यते ोे ा ात्या ा ोे ।
> ात्य ा। ाो पा ˘क्र ा ात याप्ये॑ ोे ा ा॒ ा।...
>
> एत ाात। तयो क्र ा ाोे ा ा ब ोा यते। ततो बा ॑ ा ा ा ।
> ा ो॑ पा ोे ए ा त ˘ ा य पा ˘ ा ा ात त ोे ा ा ा ष ा यात।
> तयो ाप ा तत ा त्य ाध्य ा ए ा ाक्रोच्च ा। ाक्रोच्च ा ाटो ा पा ता ए ा
> ा ा ा । त ाप ा त्यक्र ा ा ा ॅ ा ा ष ा यात, ाक्र ा य
> ˘ याया ा ˘ ात। ाक्रोच्च ा ा ा याप्ये ˘ ा ा ोे ा क्र ा ा ा ाय ा ˘ ा ा-
> ा ाप ाक्रोच्च ा ˘ या ा ˘ ा ˘ ा षात ए ा ोतयो ाप। ायो प ा

त ा षा ा ाॅ ा ा ाा ाष ाा ात।[29]

> The latitudinal motion is said to be due to that of the *śīghrocca*. How is this appropriate? Isn't the latitudinal motion of a body dependent on the motion of that body only, and not on the motion of something else? The latitudinal motion of one body cannot be obtained as being due to the motion of another. Hence [we should conclude that] Mercury goes around its own orbit in 88 days ... However this also is not appropriate because we see it going around [the Earth] in one year and not in 88 days. True, the period in which Mercury completes one full revolution around the *bhagola* (the celestial sphere) is one year only [like the Sun] ...
>
> All this can be explained thus: Their [Mercury and Venus] orbits do not circumscribe the Earth. The Earth is always outside their orbit. Since their orbit is always confined to one side of the [geocentric] celestial sphere, in completing one revolution they do not go around the twelve signs (*rāśis*). Even for them in reality the mean Sun is the *śīghrocca*. It is only their own revolutions which are stated to be the revolutions of the *śīghrocca* [in *Āryabhaṭīya*]. It is only due to the revolution of the Sun [around the Earth] that they (i.e. the interior planets, Mercury and Venus) complete their movement around the twelve signs [and complete their revolution of the Earth]. Because the *śīghra* epicycle is larger than their orbit, their orbit is completed on one side of the *śīghra* epicycle. Just as in the case of Jupiter etc. [the exterior planets] the *śīghrocca* attracts [and drags around] the *manda*-orbits on which they move (the *manda-kakṣyā-maṇḍala*), in the same way it does for these [interior] planets also. And it is owing to this attraction that these [interior planets] move around the twelve signs.

There is also a later work of unknown authorship, *Vikṣepagolavāsanā*, which confirms that it was indeed Nīlakaṇṭha who proposed that the *manda* correction for the interior planets, Mercury and Venus, should be applied to the mean planets themselves and not to their *śīghrocca*, in order to arrive at a coherent and unified theory of planetary latitudes. The relevant verses of this work are the following:

पू ाा ाये त ाॅ ाप ा पा धा ाा ा या ा
ाा ो ते ा ाॅ ऽाप ा ा ा ाध्य ा ाध्य प्रा ा।
ा ेचो ाा ाध्या ा त ा ाप ाती ा ेचे
ा त्येत ा य त ा ाती ाा ा ा या ात याात॥

ा त्या ा ा ा प ा ता ा ाा ात ा ा ा ाात ा ती
त ाात ाा ये ा ाॅ ाा ा ाता ातयो ाध्य ा ााय ाध्य ा।
प्रो ाा ा ा ा प्रा ाता ा तयो ााय या ाा ा
ा ाा मध्य ा ाात ा ा ा य याध तो ा य ाा।
ाध्योचे त ता ााप ा ा ाा ा ायत ा पते ा ा ाा ा ा॥[30]

> Indeed by the earlier *ācāryas*, even in the *manda* procedure, orbits [for Mercury and Venus] were stated by measuring them in terms of the orbit of the mean Sun, and hence for them their own mean position would be that of the mean Sun. For obtaining the latitudinal deflection (*kṣepanītau*) [of the planet] they were also applying the *manda*-correction (*mṛduphala*)—obtained by subtracting the *mandocca* [of the planet] from the mean

[29] {ABB 1957}, pp. 8-9.

[30] *Vikṣepagola-vāsanā* in {GVV 1979}, p. 52. As we shall see later, these verses closely follow the verses of Nīlakaṇṭha's *Grahassphuṭānayane vikṣepavāsanā*, in {GVV 1979}, p. 58.

Sun—to the *śīghrocca*. There is no rationale for this and that it was omitted in *Mānasa* (*Laghumānasa* of Mañjulācārya) seems quite reasonable. This approach followed by the earlier *ācāryas* is also inappropriate because the quantities [that which is used for finding the *mṛduphala* and that to which *mṛduphala* is applied] belong to different classes (*bhinnajāti*).

Therefore, it was proposed by Gārgya (Nīlakaṇṭha) that in the *manda* procedure it is their own mean position [and not the mean Sun] that should be considered as the mean position of Mercury and Venus. The dimension of *mandavṛtta* should also be taken to be given in terms of the measure of their own orbits (*svīyakakṣyā-kalābhiḥ*). In the *śīghra* process, since the orbit of the Sun is larger than their own mean orbit (*madhyavṛtta*), he also proposed that a simple way of formulating the correction would be by supposing that the mean and the *ucca* (*śīghrocca*) and their corresponding orbits (*kakṣyāvṛtta* and *śighravṛtta*) are indeed reversed.

F.5 Geometrical picture of planetary motion according to Nīlakaṇṭha

In his *Āryabhaṭīya-bhāṣya*, while commenting on verses 17–21 of the *Kālakriyāpāda*, Nīlakaṇṭha explains that the orbits of the planets, and the locations of various concentric and eccentric circles or epicycles associated with the *manda* and *śīghra* processes, are to be inferred from the computational scheme for calculating the true geocentric longitude (*sphuṭa-graha*) and the latitude of the planets (*vikṣepa*).

ता ता ा ा ा ा प ा च्च य पा ध य ा प्र ा ा त ा ा ता ा पा ध या ा -
ा ा ा प्र ा प ा त य ा ा त त्येतत ा ा पा य ा ा ा ा-
ा ा च्च ा ा त ा ा ा।

We have explained that in the case of the *tārā-graha*s (the five planets) there are two *uccas* and two epicycles. There, issues such as which epicycle has a centre on the concentric and where the other epicycle is located, can be settled by (analysing) the procedure for finding out the true longitude and latitude of the planet.

F.5.1 Geometrical picture of the motion of the exterior planets

Nīlakaṇṭha first gives the following general outline of the geometrical picture of planetary motion:

ाय ा ा ा ध । या ा ए ा ा प ा धे ा प ा ा तत्पा धौ ा ो च्चा-
ा तप्र ा ा पा ध ा ा ए ा पा धौ प ा ा च्चप्र ा प्रात ा ा ा तच्च
प्रात ा ा ा ा ा याया ा ा ा ा ा यो ा त य ा ता ा ा ा बिम्बा ा

ा ायो ा ा ाात ॎ ाताात ा य ो ा तत ॅ या ा ॅ ाॎपा धौ च्प्र ो ॅ ॅ ति
पा ॅ ा ाय ा। ता ाप ॅ ा ा ॅ ा ॅ ा याये ा ा ोष्य पा ॅ ा ाय ा।[31]

Here, what is intended to be conveyed is as follows: The centre of the *kakṣyā-maṇḍala* (concentric) is also the centre of the *śīghra* epicycle; on that epicycle, at the location of the *śīghrocca*, is the centre of the *manda* epicycle; in the same way, on that *manda* epicycle at the location of *mandocca* is the centre of the *pratimaṇḍala* (eccentric). (The circumference of) that *pratimaṇḍala* is equal to the circumference of the sky (*ākāśa-kakṣyā*) divided by the revolution number of the planet. The planetary orb moves with the same linear velocity, as that of the others, in that (*pratimaṇḍala*) only. The corresponding concentric (*kakṣyā-maṇḍala*) should be drawn with the same dimension with its centre on the *śīghra* epicycle at the location of *śīghrocca*. There also the circle of the hypotenuse is to be obtained by the process of iteration as per the rule for the *manda-karṇa*.

Later, while commenting on verse 3 of *Golapāda*, Nīlakaṇṭha explains how the above picture needs to be modified when the latitudinal motion is also taken into account. The main feature is that it is the *manda* epicycle together with the eccentric which is inclined to the ecliptic and not the *śīghra* epicycle (which represents the Earth–Sun relative motion):

ा ो ॅ ाध्य ाा ा ॅ य ाॅत्ये ा प ा ॅ ॅ ाा ा ा य ाॎ ा ा य पा धौ य ाॎोच्च-
ा ाप्र ा ता ा ॅ ाा ा ॅ या ा ॅ ॅ ॅ ा ाात ा ा यापा ॅ ऐ ो ॅ ॅ ा।
त ा ा ोच्च ाा ा ा ा य ॅ ा ा ॅ ॅ य ा ॅ ॅ ॅ ा। ऐ ा ोता ा ााा ा ॅ ाा ा
प ा ॅ ॅ ाा ा ाा ात ध ा ॅ ा तो ा ा ात ा ा ाा ा ााा ा।[32]

It has already been stated in the *Kālakriyāpāda* that on the *śīghra-vṛtta*, which has its centre at the centre of the celestial sphere and is in the plane of the ecliptic, the point which corresponds to the *śīghrocca* is in fact the centre of the *kakṣyā-maṇḍala* (concentric) in the *manda* process. The same (*śīghrocca*) is also the centre of the *manda-nicocca-vṛtta* (the *manda* epicycle) and also of the (*manda*) *karṇa-maṇḍala* (the hypotenuse circle or the orbit). In this way these three circles (*manda* concentric, epicycle and hypotenuse circle) are inclined to the ecliptic towards both the north and the south.

Based on the description presented above, we arrive at the geometrical picture of motion—for an exterior planet—as shown in Fig. F.8*a*. In this figure, O represents the location of the observer and is considered to be the *bhagola-madhya* (the centre of the celestial sphere). The circle centred around O, with radius equal to the tabulated radius of the *śīghra* epicycle, r_s, is called the *śīghra-nīcocca-vṛtta*, on which the *śīghrocca* or the mean Sun S is located.

It is said that the *manda-nīcocca-vṛtta* (also called the *mandaparidhi*) is a circle with the *śīghrocca* as the centre. The *mandocca* U is located on this circle, whose radius is equal to the (variable) radius of the *manda* epicycle. The *pratimaṇḍala* on which the planet P moves is centred at the *mandocca*. SP is the *manda-karṇa* denoted by K and $\Gamma\hat{S}P$ is the *manda-sphuṭa*. $\Gamma\hat{O}P$ is the true geocentric planet known as the *śīghra-sphuṭa*. The distance of the planet from the centre of the *bhagola* is denoted by K_s and it is also the *śīghra-karṇa*.

Among the various circles depicted in Fig. F.8*a*, it is said that the circles centred around the *śīghrocca* S, namely the *manda-nīcocca-vṛtta*, the *manda-karṇa-vṛtta*

[31] {ABB 1931}, vol. II, p. 70.

[32] {ABB 1957}, p. 5.

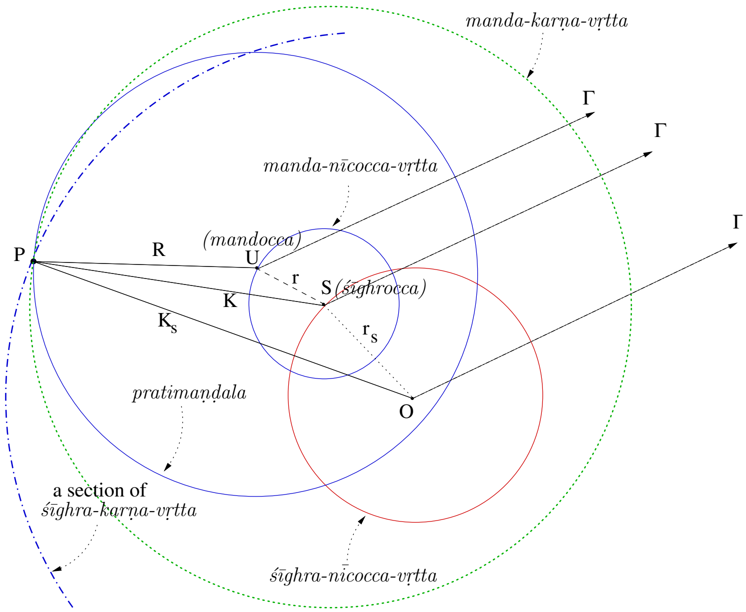

Fig. F.8*a* Geometrical picture of the motion of an exterior planet given by Nīlakaṇṭha.

and the *manda* concentric (which is not indicated in the figure), are inclined to the plane of the ecliptic towards the north and the south. The figure also depicts a section of the *śīghra-karṇa-vṛtta*—centred around *O*—which represents the instantaneous orbit (the orbit in which the planet moves at that instant) of the planet with respect to the Earth.

F.5.2 Geometrical picture of the motion of the interior planets

Nīlakaṇṭha explains in the commentary on verse 3 of *Golapāda* that the above geometrical picture of motion needs to be modified in the case of the interior planets. We have earlier (in Section F.4.4) cited a part of this discussion where Nīlakaṇṭha had noted that the interior planets go around the Sun in orbits that do not circumscribe the Earth, in a period that corresponds to the period of their latitudinal motion, and that they go around the zodiac in one year as they are dragged around the Earth by the Sun. Having identified the special feature of the orbits of the interior planets that they do not circumscribe the Earth, Nīlakaṇṭha explains that it is their own orbit, which is smaller than the *śīghra-nīcocca-vṛtta*, that is tabulated as the epicycle in a measure where the latter is 360 degrees.

ते ाे ो ाप ा ो यत्य य ˉ प्येते। तयो ाा्रोच्च ाा ा ा ा य षा ात ाया ो ो ा
पा ˉ ााय ा ˉ या ा ा ो ा ाा्रोच्च ाा ा ा ाते ो ा ˉ प्यते। ा ोच्च ाा ा ा ा ा त ो ो ा
पा ˉ ााय पा त ा।[33]

These two circles (the concentric and the *śīghra* epicycle) are now to be imagined in the contrary way. Of them, the concentric itself (being smaller than the epicycle) is given in units where the *śīghra* epicycle is taken to be 360°, and will now play the role of epicycle. The *manda* epicycle is also taken to be tabulated in terms of this (concentric).

Nīlakaṇṭha then goes on to explain the process of computation of the true longitude of these planets in the same manner as outlined in *Tantrasaṅgraha* and one that corresponds to the following geometrical picture of motion.

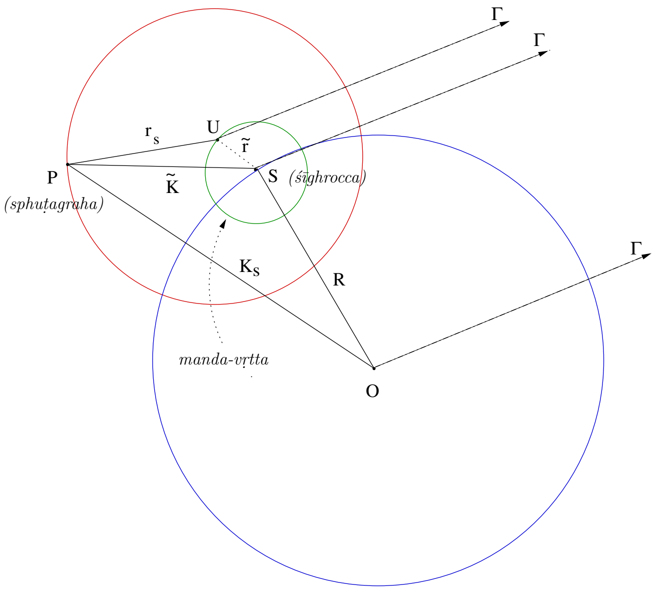

Fig. F.8*b* Geometrical picture of the motion of an interior planet given by Nīlakaṇṭha.

The geometrical picture of the motion of the interior planets as presented by Nīlakaṇṭha is shown in Fig. F.8*b*. Here, O is the observer, assumed to be at the centre of the celestial sphere (the *bhagola-madhya*). S is the *śīghrocca* which is taken to be the mean Sun for all the planets. P is the planet moving around the mean Sun in an eccentric orbit. This eccentric orbit is centred at U, the *mandocca*. The point U itself is conceived to be moving on the *manda-nīcocca-vṛtta* centred around S.

For interior planets the planet–Sun distance is smaller than the Earth–Sun distance. Hence, the radius of the planet's eccentric orbit (UP) is taken to be the radius of the *śīghrocca-nīca-vṛtta* r_s, and the radius of the mean Sun's orbit (OS) is taken

[33] {ABB 1957}, p. 9.

to be the *trijyā*, R. Further, since the *mandapratimaṇḍala*, or the *manda* eccentric on which the planet moves, is of dimension r_s and not R, the (variable) *manda* epicycle r itself is to be scaled by a factor $\frac{r_s}{R}$ and will be $\tilde{r} = r\frac{r_s}{R}$. Correspondingly the (iterated) *manda-karṇa* K will also be scaled to $\tilde{K} = K\frac{r_s}{R}$.

Nīlakaṇṭha presents a clear and succinct statement of the geometrical picture of planetary motion for both interior and exterior planets in both *Golasāra* and *Siddhānta-darpaṇa*. The verses from the latter are cited below:

ा ा ा ा ाा ा ा ¯त्यच्च ाता याप।
ा ाोत ` ¯ ो ा ा ा ाध्य ााा ा ¯ ा॥
ाध्या ¯ाात ाा येषा त मध्य ाा ा ा ा ा।
तेषा ैय ा ा ा ा ा ाा ा ा ाो ¯ ाध्य ा ा॥
ैयतो ा त ै ा प्र ाायो ` ा ा यो ।
ा ा ा य ैाा ा ाय ा ा ा ¯ ा ात॥[34]

The [eccentric] orbits on which planets move (the *graha-bhramaṇa-vṛtta*) themselves move at the same rate as the apsides (the *ucca-gati*) on the *manda-vṛtta* [or the *manda* epicycle drawn with its centre coinciding with the centre of the *manda* concentric]. In the case of the Sun and the Moon, the centre of the Earth is the centre of this *manda-vṛtta*.

For the others [namely the planets Mercury, Venus, Mars, Jupiter and Saturn] the centre of the *manda-vṛtta* moves at the same rate as the mean Sun (*madhyārka-gati*) on the *śīghra-vṛtta* [or the *śīghra* epicycle drawn with its centre coinciding with the centre of the *śīghra concentric*. The *śīghra-vṛtta* for these planets is not inclined with respect to the ecliptic and has the centre of the celestial sphere as its centre.

In the case of Mercury and Venus, the dimension of the *śīghra-vṛtta* is taken to be that of the concentric and the dimensions [of the epicycles] mentioned are of their own orbits. The *manda-vṛtta* [and hence the *manda* epicycle of all the planets] undergoes increase and decrease in size in the same way as the *karṇa* [or the hypotenuse or the distance of the planet from the centre of the *manda* concentric].

As was noted earlier, the renowned Malayalam work *Gaṇita-yukti-bhāṣā* (c. 1530) of Jyeṣṭhadeva also gives a detailed exposition of the above geometrical picture planetary motion. The expressions for the longitudes for the exterior and interior planets obtained from the above pictures are essentially the same as the ones in the Keplerian model in (F.46) and (F.50).

F.6 Nīlakaṇṭha's cosmological model

While discussing the geometrical picture of planetary motion, *Āryabhaṭīya-bhāṣya* as well as *Golasāra* and *Siddhānta-darpaṇa* consider the orbit of each of the planets individually, and they are not put together in a single cosmological model of the planetary system.

There is of course a remarkable passage in *Āryabhaṭīya-bhāṣya* (which we have cited earlier (see Section F.4.4) while explaining Nīlakaṇṭha's rationale for the revision of the traditional planetary model) where Nīlakaṇṭha explains that the Earth

[34] {SDA 1978}, p. 18.

is not circumscribed by the orbit of the interior planets, Mercury and Venus; and that the mean period of motion in longitude of these planets around the Earth is the same as that of the Sun, precisely because they are being carried around the Earth by the Sun. In fact, Nīlakaṇṭha seems to be the first savant in the history of astronomy to clearly deduce from his computational scheme (and not from any speculative or cosmological argument) that the interior planets go around the Sun and that the period of their motion around the Sun is also the period of their latitudinal motion.

एत ाात। तयो ्र ा ा ा ो ा ा ू ब ेा यते। ततो बा ेा ा ा ू ।
ा ाे ै पा ो ए ा त ा य पा ा ाा ा ाात त ा ो ा ा ा ा ाा ाष ाा यात।
तयो ाप ा तत ाा त्य ाध्य ा ए ा ाा्रोच्च ा। ाा्रोच्च ा ा ाेा ो ा पा ता ए ा
ा ा ा ाा । त ााप ाा त्य्र ा ा ा ाा ा ा ा ाा ाष ाा यात, ाा्र ा ा य
यायाा ा ाात। ाा्रोच्च ाा ा ा ा याप्ये ाा ा ा ो ा ा्र ा ा ा ा ा। य ाा ाा-
ा ाा ााप ाा्रोच्च ा ा या ा ाा ाा षात ए ा ोतयो ाप। ायो प ा
त ा ष ा ा ाा ा ा ा ाा ाष ाा ात।

Nīlakaṇṭha presents his cosmological model very clearly in a remarkable short tract called *Grahasphuṭānayane vikṣepavāsanā*, which seems to have been written after *Āryabhaṭīya-bhāṣya* as it quotes extensively from it. Here he clearly integrates the geometrical picture of motion of different planets into a single model of the planetary system by identifying the *śīghrocca*, that each of the planets goes around, with the physical 'mean Sun moving on the orbit of the Sun'. Based on this identification, Nīlakaṇṭha also states that the ratio of the radius of the *śīghra* epicycle to that of the concentric is nothing but the ratio of the mean radius of the orbit of the Sun around the Earth to the mean radius of the orbit of the planet itself, in the case of the exterior planets, while it is the other way around in the case of the interior planets. He further explains that this difference between the exterior and interior planets is because, in the case of the interior planets, their orbit is smaller than the orbit of the Sun around the Earth and the dimensions of the epicycle and concentric have to be interchanged. In Nīlakaṇṭha's own words:

ा ा ापात यत ा ाा ा ध ा ाा त ा ाात
ा ाा ा ाा ाा या ा ाता ा ा ो ा ा ा त य ाै ।
तो ो ाा या ाप ा ा य ाध्य ा ा ा ाा-
ाा ा या ा ो ा ाता ा या ा ाध्या ा ॥

ा ाा ाा ाा ाा ा ा ाात या ााप ायता या
ो्रा य ाा ा ा ााा ा बधा ातयो ा या ाा ा ।
ाा या ााप ायता प ाा ा ाते ाा्र ा ो यतोऽतो
ाा ाो ाध्य ा ाध्य ा ाात ा ा ाधी ााय ाध्य ा ोच्च ा॥

पू ाा ाायै त ाा ाप ा पाधा ाा ा या ाा ा
ााो ते ा ाा ऽाप ा ा ा ाध्य ा ाध्य तयो यात।
ा ोच्चो ाा ाध्या ा त ा ोप ााती ा ोच्चे

¯ा त्या ा ा ा प ो ता ा ाा ाता ा ा ा ाातदा ेतो ॥

ाा े ा ाध्य ो ाात ¯ ा ाा ाधो ाध्य ा ¯ प ााय
ाा ा ाा ा ा ा प्रा ाता ा तयो ााय ¯ या ¯ ाा ा ।
ोे ाा मध्य ा ाात ा ा ¯¯ा य याध दाा ा य
ाध्योच्चे त ता ाााप ा ा ाा ा ायत ¯ प ाये ा य ाा॥[35]

The *manda-vṛttas* of the Moon and the others (the five planets) are deflected from the two nodes of their own orbits, half-way towards the north and the south of the ecliptic (*krānti-vṛtta*) by a measure that has been specified separately [for each planet] and which remains the same for all times. There [again] the *manda-vṛtta* of the Moon is centred at the centre of the ecliptic (*apamavalaya*), whereas the *manda-vṛttas* of Mars etc. (the five planets) are centred at the mean Sun which lies on the orbit of the Sun (*dinkara-kakṣyāstha-madhyārka*) situated in the celestial sphere (*bhagola*).

Moreover, in the case of Mars, Jupiter and Saturn, the [dimensions of their] *śīghra-vṛttas* have been stated by measuring the orbit of the [mean] Sun (*arka-kakṣyā*) in terms of minutes of (the dimensions of) their own orbits (*nija-vṛti-kalayā*). However in the case of Mercury and Venus, the [dimensions of their] *śīghra-vṛttas* have indeed (*punaḥ*) been stated by measuring their own orbits in terms of the minutes of (the dimension of) the orbit of the [mean] Sun (*arka-kakṣhyā-kalābhiḥ*). Since it is done this way (*yataḥ*), (*ataḥ*) the mean Sun becomes the mean planet in the *śīghra* procedure (*calavidhi*) and their own mean positions become the *śīghroccas* (*caloccas*).

Indeed, by the earlier *acāryas*, even in the *manda* procedure [their own] orbits [for Mercury and Venus] were stated by measuring them in terms of the orbit of the mean Sun, and hence for them their own mean position would be that of the mean Sun. Even in this school (*asmin hi pakṣe*) for obtaining the latitudinal deflection (*kṣepanītau*) [of the planet] they were applying the *manda* correction (*mṛduphala*) [which was] obtained by subtracting the *mandocca* [of the planet] from the mean Sun, to the *śīghrocca*. This is however inappropriate because these (the quantity used for finding the *mṛduphala* and the quantity to which the *mṛduphala* is applied) belong to different classes (*bhinnajāti*).

Therefore, even in the *manda* procedure it is their own mean position [and not the mean Sun] that should be considered as the mean position of Mercury and Venus. The dimension of the *manda-vṛtta* should also be taken to be given in terms of the measure of their own orbits (*svīya-kakṣyā-kalābhiḥ*). In the *śīghra* process, since the orbit of the Sun is larger than their own mean orbit (*madhyavṛtta*), one has to devise an intelligent scheme (*yuktyā*), in which the mean and the *ucca* (*śīghrocca*) and their corresponding orbits (*kakṣyā-vṛtta* and *śīghra-vṛtta*) are reversed.

The first verse clearly describes the cosmological model of Nīlakaṇṭha, which is that the five planets, Mercury, Venus, Mars, Jupiter and Saturn, go around the mean Sun in an eccentric orbit—inclined to the ecliptic (see Fig. F.9)—while the mean Sun itself goes around the Earth[36]. It is in the second verse that Nīlakaṇṭha makes the remarkable identification that

$$\frac{r_s}{R} = \frac{\text{mean Earth–Sun distance}}{\text{mean Sun–planet distance}} \qquad \text{(for exterior planets)} \qquad \text{(F.29}a\text{)}$$

35 {GVV 1979} 1979, p. 58. As we noted earlier, the initial verses of the anonymous tract *Vikṣepagolavāsanā* closely follow the above verses of Nīlakaṇṭha.

36 As we noted earlier, this cosmological model is the same as the one proposed by Tycho Brahe, albeit on entirely different considerations, towards the end of sixteenth century.

$$\frac{r_s}{R} = \frac{\text{mean Sun–planet distance}}{\text{mean Earth–Sun distance}} \quad \text{(for interior planets).} \qquad \text{(F.29}b\text{)}$$

where r_s is the radius of the *śīghra* epicycle and R is the radius of the concentric. We had noted earlier in Section F.2 that the *śīghra*-process serves to transform the heliocentric longitudes to geocentric longitudes, precisely because the above relations (F.29*a*) and (F.29*b*) are indeed satisfied (see Table F.3), even though the traditional Indian astronomical texts did not conceive of any such relation between the radii of the *śīghra* epicycles and the mean ratios of Earth–Sun and Sun–planet distances. In fact, Nīlakaṇṭha seems to be the first Indian astronomer to explicitly state the relations (F.29*a* and F.29*b*), which seems to follow clearly from his identification of the *śīghrocca* of each planet with the physical 'mean Sun lying on the orbit of the Sun' (*dinakara-kakṣyāstha-madhyārka*).[37]

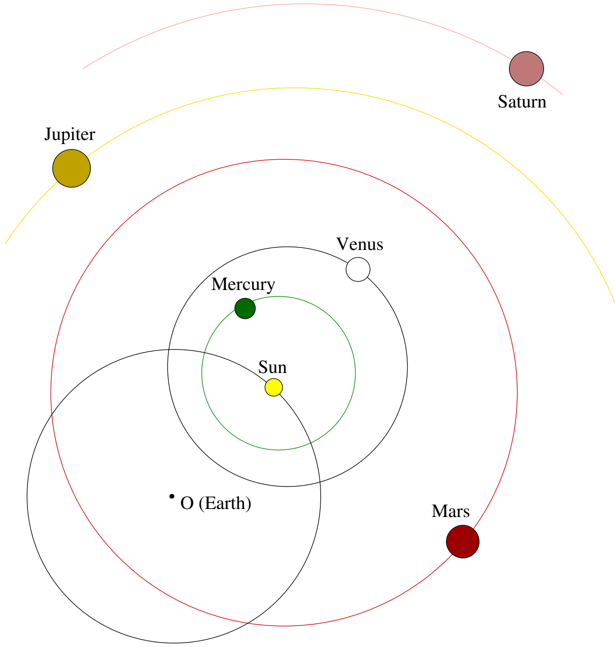

Fig. F.9 Nīlakaṇṭha's cosmological model showing the five planets moving in eccentric orbits around the mean Sun.

The last two verses above discuss the rationale behind the revised planetary model proposed by Nīlakaṇṭha and have been dealt with already in Section F.4.4. However, what is noteworthy in the context of the cosmological model of

[37] As we noted earlier, Nicholas Copernicus also seems to have arrived at the same relation (perhaps around the same time as Nīlakaṇṭha) by identifying the epicycle associated with the so-called 'solar anomaly' in the Ptolemaic model with the orbit of the Earth around the Sun in the case of the exterior planets and with the orbit of the planet itself in the case of the interior planets.

Nīlakaṇṭhais the clear statement that is found again in these verses that the orbits of the interior planets are indeed smaller than the orbit of the Sun (*dinakaravalaya*).

F.7 The problem of planetary distances

F.7.1 Planetary distances in traditional Indian astronomy

Unlike the longitudes and latitudes of planets, the planetary distances were not directly amenable to observation in ancient astronomy and their discussion was often based upon some speculative hypothesis. In traditional Indian planetary theory, at least from the time of Āryabhaṭa, the mean planetary distances were obtained based on the hypothesis that all the planets go around the Earth with the same linear velocity—i.e. they all cover the same physical distance in a given period of time.

Āryabhaṭa, indicates this principle in verse 6 of *Gītikāpāda* of *Āryabhaṭīya*, where he also mentions that one minute of arc in the orbit of the Moon measures 10 *yojanas* (which is a distance measure used in Indian Astronomy). In verse 7 of *Gītikāpāda* he gives the diameters of the Earth, Moon and the Sun in *yojanas*. The number of revolutions of the various planets (see Table F.1) are given in verses 3 and 4 of *Gītikāpāda*. Based on these, we can work out the *kakṣyā* (mean orbital circumference) and the *kakṣyāvyāsārdha* (orbital radii) of the Sun, Moon and the various planets as given in Table F. 4.

Planet	Diameter (*yojanas*)	Revolutions in a *Mahāyuga*	*Kakṣyā* (circumference) (in *yojanas*)	*Kakṣyāvyāsārdha* (radius) (in *yojanas*)	Radius/Earth-diameter
Earth	1050				
Moon	315	57753336	216000	34380	65.5
Sun	4410	4320000	2887667	459620	875.5

Table F.4 *Kakṣyāvyāsārdhas* (orbital radii) of the Sun and the Moon given by Āryabhaṭa.

From Table F.4, we can see that the mean distance of the Moon has been estimated by the Indian astronomers fairly accurately (the modern value of the mean distance of Moon is about 60 Earth radii), but the estimate of the distance of Sun is short by a factor of around 30 (the modern value of the mean distance of Sun is around 23500 Earth radii).[38]

[38] The ancient astronomers' estimates of the Earth–Sun distance were all greatly off the mark. Ptolemy estimated the mean distance of the Sun to be 1210 Earth radii which is low by a factor of 20. The values given by Copernicus and Tycho were also of the same order. The value estimated by Kepler was short by a factor of 6. In 1672 the French astronomer Cassini arrived at a value which is within 10% of the actual mean distance.

Planet	Diameter (*yojanas*)	Revolutions in a *Mahāyuga*	*Kakṣyā* (circumference (in *yojanas*))
Moon	5,77,53,336	2,16,000	34,380
Sun	43,20,000	28,87,667	4,59,620
Mercury	1,79,37,020	6,95,473	1,10,696
Venus	70,22,388	17,76,421	2,82,747
Mars	22,96,824	54,31,195	8,64,481
Jupiter	3,64,224	3,42,50,133	54,51,480
Saturn	1,46,564	8,51,14,493	1,35,47,390

Table F.5 *Kakṣyāvyāsārdhas* (orbital radii) of the planets given by Āryabhaṭa.

The *kakṣyāvyāsārdhas* given in Table F.5 give the mean Earth–planet distance as per the planetary model of Āryabhaṭa. They essentially served the purpose of fixing the order of the various planets,[39] which is given by Āryabhaṭa in the *Kālakriyāpāda* of *Āryabhaṭīya*:

ा ा ाध ाै ा ा ा ौा ा ा बध ा ा ।
एषा ाध ा ूा ा ाेधा ूता ा ाध्य ा॥[40]

Below the stars [are the orbits of] Saturn, Jupiter, Mars, Sun, Venus, Mercury and Moon. Below them is the solid Earth [suspended] in the middle of the space.

Āryabhaṭa gives a prescription for the true Earth–planet distance towards the end of *Kālakriyāpāda*:

ूताा ा ा ा ा या ाध त ा ा ा ा ा ।[41]

The Earth–planet distance is given by the product of the [*manda* and *śīghra*] *karṇas* of the planet divided by the radius (of the concentric).

Thus the prescription of Āryabhaṭa is that

$$\text{Earth–planet distance} = \frac{\textit{manda-karṇa} \times \textit{śīghra-karṇa}}{R}. \qquad \text{(F.30)}$$

Nīlakaṇṭha in his *Āryabhaṭīya-bhāṣya* explains that, since usually the *śīghra-karṇa* is evaluated with respect to a concentric of the standard radius, the above prescription of Āryabhaṭa implies that the Earth–planet distance is actually given by the *śīghra-karṇa* which is evaluated with respect to a concentric circle whose radius is given by the (iterated) *manda-karṇa*.[42] This is in accordance with his geometrical picture of planetary motion as given, say, in Fig. F.6.

---

[39] On the other hand, in the early Greco-European tradition, there was considerable ambiguity concerning the order of planets. Neither does Ptolemy discuss the issue of planetary distances in his *Almagest*. In his later work, *Planetary Hypothesis*, Ptolemy uses the principle that the orbit of each planet fills the entire space between those of the neighbouring planets to arrive at estimates of planetary distances.

[40] {AB 1976}, p. 102.

[41] {AB 1976}, p. 111.

[42] {ABB 1931}, pp. 53–4.

The above relation (F.30) gives the true Earth–planet distance in minutes, as usually the *manda-karṇa* and *śīghra-karṇa* are evaluated with respect to a concentric circle whose radius is given by the *trijyā*, $R \approx 3438'$. From this, the true Earth–planet distance (sometimes called the *sphuṭa-kakṣyā*) in *yojanas* is obtained by using the relation

$$\textit{Sphuṭa-kakṣyā}\ (\text{in } yjn) = \frac{\text{Earth–planet distance (in min)} \times \textit{kakṣyā-vyāsārdha}\ (\text{in } yjn)}{\text{Radius (in min)}}. \qquad \text{(F.31)}$$

The above relation is based on the hypothesis employed in the traditional Indian planetary theory that the *kakṣyāvyāsārdha* given in Table F.5 represents the mean Earth–planet distance in *yojanas*.

F.7.2 Nīlakaṇṭha *on planetary distances*

In the fourth chapter of *Tantrasaṅgraha*, dealing with lunar eclipses, Nīlakaṇṭha gives the mean radius of the orbit of the Moon in *yojanas* to be the *trijyā* (radius) in minutes multiplied by 10, i.e. 34380 *yojanas*. He also states that the radii of the orbits of the Sun and the Moon are in inverse proportion to their *bhagaṇas*, or the number of revolutions in a *Mahāyuga*. He further gives the diameters of the Moon and Sun in *yojanas* to be 315 and 4410, respectively, and also states that the diameter of the Earth is to be found from the circumference of 3,300 *yojanas* given in verse 1.29. Table F. 6 gives diameters and mean distances in *yojanas*.

Planet	Diameter (*yojanas*)	Revolutions in a *Mahāyuga*	*Kakṣyā* (circumference) (in *yojanas*)	*Kakṣyā-vyāsārdha* (radius) (in *yojanas*)	Radius/Earth-diameter
Earth	1050.4				
Moon	315	5,77,53,336	216,000	34,380	65.5
Sun	4410	43,20,000	28,87,667	4,59,620	875.5

Table F.6 *Kakṣyāvyāsārdhas* (orbital radii) of the Sun and the Moon given by Nīlakaṇṭha.

Nīlakaṇṭha then states that the *sphuṭa-yojana-karṇas*, the first approximations to the true distance of the centres of Sun and Moon from the centre of the Earth, are given by their mean distances multiplied by the iterated *manda-karṇa* divided by the radius. Finally he gives the *dvitīya-sphuṭa-yojana-karṇas*, the true distances taking into account the second correction, corresponding to the so-called evection term, for both Sun and Moon at times of conjunction and opposition. The general expression for *dvitīya-sphuṭa-yojana-karṇa* is given in the first two verses of Chapter

8. *Tantrasaṅgraha* does not discuss the corresponding geometrical picture of lunar motion, which is however dealt with in detail in *Yuktibhāṣā*[43].

Nīlakaṇṭha takes up the issue of planetary distances towards the very end of the last chapter (Chapter 8) of *Tantrasaṅgraha*. Here, he first notes that the mean radius of the orbit (*kakṣyāvyāsārdha*) of each planet is to be found in the same way as was prescribed in the case of the Sun in Chapter 4, namely by multiplying the *kakṣyāvyāsārdha* and the revolutions in a *Mahāyuga* of the Moon, and dividing the product by the revolutions of the planet in a *Mahāyuga*.

ा ा ाद्य ̄ ̄ याया ेया येषा ा ाा तत ।[44]

This is essentially the principle of traditional Indian astronomy that all the planets travel equal distances in their orbits in any given period of time, or that they all have the same linear velocity. Nīlakaṇṭha in fact states this principle explicitly in his *Siddhānta-darpaṇa* as follows:

ा यो ा ा ाा या ाे े ̄ ा ाात ।[45]

The velocity in minutes [per unit time] (*kalāgati*) of the Moon multiplied by 10 is the velocity of [each] planet in *yojanas* [per unit time] (*yojanabhukti*).

Based on the number of revolutions given in Chapter 1 of *Tantrasaṅgraha* we can calculate the mean orbital radii (*kakṣyāvyāsārdha*) of all the planets as given in Table F.7.

Planet	Revolutions in a *Mahāyuga*	*Kakṣyā* (circumference in *yojanas*)	*Kakṣyāvyāsārdha* (radius in *yojanas*)
Moon	57753320	216000	34380
Sun	4320000	2887666	459620
Mercury	17937048	695472	110696
Venus	7022268	1776451	282752
Mars	2296864	5431195	864465
Jupiter	364180	34254262	5452137
Saturn	146612	85086604	13542951

Table F.7 *Kakṣyāvyāsārdhas* (orbital radii) of the planets given by Nīlakaṇṭha.

While the values of the *kakṣyāvyāsārdha* given by Nīlakaṇṭha differ only marginally from those given in *Āryabhaṭīya* (see Table F.5), Nīlakaṇṭha's inter-

[43] {GYB 2008}, Section 11.36, pp. 584–7, 786–8, 975–80. It may be of interest to note that the maximum variation in the distance of Moon due to the second correction in Nīlakaṇṭha's model is only of the order of 10% and not the ridiculous figure of around 50% found in the Ptolemaic model of evection. Of course, the expression for the second correction given by Nīlakaṇṭha is essentially the same as the one given by Mañjulācārya (c. 932) and is more accurate and elegant than the Ptolemaic formulation of evection. See also M. S. Sriram, Planetary and Lunar Models in *Tantrasaṅgraha* and *Gaṇita-Yuktibhāṣā*, in *Studies in History of Indian Mathematics*, ed. by C. S. Seshadri, Hindustan Book Agency, New Delhi 2010, pp. 353–89.

[44] {TS 1958}, p. 154.

[45] {SDA 1976}, p. 13.

pretation of this *kakṣyāvyāsārdha* and his prescription for the true Earth-planet distance in *yojanas* (the *sphuṭa-kakṣyā*) are indeed very different from what we outlined earlier in connection with the traditional Indian planetary model. Nīlakaṇṭha presents his prescription for *sphuṭa-kakṣyā* rather tritely in just a single verse of *Tantrasaṅgraha*:

ार्‌ ‾ार्‌ ‾ याया त ेि ा ा ताया ।
ा ाा ा ‾ ‾ या यात त ा म्ब ाा ा॥[46]

[In the case of Mars, Jupiter and Saturn], the mean radii of their orbits (*kakṣyāvyāsārdhas*) multiplied by the *śīghra-karṇa* [and divided by the *trijyā*] give the true orbital radii (*sphuṭa-kakṣyās*). In the case of Mercury and Venus their mean orbital radii (*kakṣyāvyāsārdha*) multiplied by the *śīghra-karṇa* and divided by the mean radii of their own orbits (*tad-vṛttas*), give the true values of their orbital radii (*sphuṭa-kakṣyās*). And from that the *lambana* etc. [must be calculated].

The above prescription has been clearly explained by Śaṅkara Vāriyar as follows:

ा तायो ‾ ‾ा ा ाात ‾ या या ाधयो ा ा ा ाा ‾ो ा ा ा त्य ाा ा ाप्रा ात-
ा ा ा या ाधे ा ा ा ोत। ता ब्धा ‾ ‾ या ा ाात। येषा त प्रा ा ब्ध
‾ या या ाध ा ाा ाा ात ा ा यया ा ा ा ा ाात ा ाोष ।[47]

In the case of Mercury and Venus, the mean radii of their orbit in *yojanas* (*kakṣyāvyāsārdha-yojana*) has to be multiplied by the *śīghra-karṇa* and divided by the radius of their own orbit which is the indeed the *śīghra-vṛtta*. The result is the true radius of the orbit (*sphuṭa-kakṣyā*) [in *yojanas*]. For the other planets (Mars, Jupiter and Saturn) the difference is that the mean radii (*kakṣyāvyāsārdhas*) [in *yojanas*] obtained as before and multiplied by their own *śīghra* [*karṇa*] should be divided by the radius of the concentric (the *trijyā*) [in order to obtain true radius of the orbit in *yojanas*].

There is a verse in *Golasāra* which seems to give a partially similar prescription for the case of interior planets:

ाा ाताा ा ा त ाि ा ा ााप ‾ा ा त्य ‾ ‾ाा य ा॥[48]

[For Mercury and Venus] their distance from the Earth (their *śīghra-karṇa*) multiplied by their (mean orbit radius in) *yojanas* is to be divided only by their last *śīghra-phala* (or the radius of the *śīghra* epicycle).

Thus, Nīlakaṇṭha's prescription for the *sphuṭa-kakṣyā* or the true Earth–planet distance in *yojanas* can be expressed as follows:

$$\textit{Sphuṭa-kakṣyā} = \frac{\textit{kakṣyāvyāsārdha} \times \textit{śīghra-karṇa}}{\text{Radius}} \qquad \text{[exterior]} \quad (F.32)$$

$$\textit{Sphuṭa-kakṣyā} = \frac{\textit{kakṣyāvyāsārdha} \times \textit{śīghra-karṇa}}{\text{Radius of } \textit{śīghra} \text{ epicycle}} \qquad \text{[interior].} \quad (F.33)$$

[46] {TS 1958}, chapter 8, verses 37b–38a.
[47] {TS 1958}, p. 155.
[48] {GS 1970}, p. 23.

The expression for the *sphuṭa-kakṣyā* for the exterior planets seems to be the same as that given by (F.31) used in the traditional planetary models, while that for the interior planets (F.33) differs by the fact that the radius (of the concentric) in the denominator in (F.31) is replaced by the radius of the *śīghra* epicycle.[49] In other words, the *kakṣyāvyāsārdha* for Nīlakaṇṭha is a mean distance in *yojanas* which corresponds to the radius of the concentric in the case of the exterior planets; and it is a mean distance in *yojanas* corresponding to the radius of the *śīghra* epicycle in the case of interior planets. If we take a careful look at the geometrical picture of planetary motion given in Fig. F.8*a* and Fig. F.8*b*, we can easily see that, according to Nīlakaṇṭha, the *kakṣyāvyāsārdha* in *yojanas* (given in Table F.7), following the equal linear velocity principle, is not the mean Earth–planet distance, but is in fact the *śīghrocca*–planet distance.

This fact that the *kakṣyāvyāsārdha* in *yojanas*, obtained based on the principle that all the planets cover equal distances in equal times, should be understood as the mean *śīghrocca*–planet distance (and not the mean Earth–planet distance) has been clearly stated by Nīlakaṇṭha in the passage from *Āryabhaṭīya-bhāṣya* that we cited earlier while discussing the geometrical picture of planetary motion:

> ¯या ा ˉ ˋ˜ ए ा ाापा धे ाप ˋ˜ ˉ।। तत्पा धौ ाोच्चा ा तप्र ˋ ो ा -
> पा ाध ˋ˜ ।। ए ा पा धौ प ा ा ोच्चप्र ˋ ो प्रात ा ˉ ˋ˜ ।। तच्च प्रात ा ˉ -
> ाा ा ा ˉयाया ा ा ा ाा ाायो ा त य ।। ता ा ो ा ा ाबम्बा ाते ा ा-
> यो ा ा ाात ा ाात।

> The centre of the *kakṣyā-maṇḍala* (concentric) is also the centre of the *śīghra* epicycle; on that epicycle, at the location of the *śīghrocca*, is the centre of the *manda* epicycle; in the same way, on that *manda* epicycle at the location of *mandocca* is the centre of the *pratimaṇḍala* (eccentric). (The circumference of) that *pratimaṇḍala* is equal to the circumference of the sky (*ākāśa-kakṣyā*) divided by the revolution number of the planet. The planetary orb moves with the same linear velocity as that of the others in that (*pratimaṇḍala*) only.

In the above passage in *Āryabhaṭīya-bhāṣya*, Nīlakaṇṭha states that the planets are orbiting with equal linear velocity in eccentric orbits about the *śīghrocca*. In other words, the *kakṣyāvyāsārdhas* in *yojanas* given in Table F.7 refer to the mean *śīghrocca*–planet distances in Nīlakaṇṭha's model. This seems to be a major departure from the conventional identification of these *kakṣyāvyāsārdhas* (derived in inverse ratio with *bhagaṇas*) with mean Earth–planet distances.

Thus, both in his *Tantrasaṅgraha* (c. 1500 CE) and in the later work *Āryabhaṭīya-bhāṣya*, Nīlakaṇṭha seems to be clearly working towards an alternative cosmology, where the planets—Mercury, Venus, Mars, Jupiter and Saturn—all go around the *śīghrocca*. His attempt to modify the traditional prescription for the planetary distances is also a step in this direction. However, even this modified prescription for the planetary distances that Nīlakaṇṭha proposes in *Tantrasaṅgraha* and

[49] This important difference between the *sphuṭa-kakṣyās* for the exterior and interior planets, in Nīlakaṇṭha's theory, seems to have been overlooked by Pingree in his analysis of 'Nīlakaṇṭha's Planetary Models' (D. Pingree, Journal of Indian Philosophy 29, 187–95, 2001). Pingree uses the *Sphuṭa-kakṣyā* formula (F.32), as applicable to the exterior planets, to arrive at the upper and lower limits of the Earth–planet distance in the case of Venus.

Āryabhaṭīya-bhāṣya is not really consistent with the cosmological model that he clearly enunciates in his later tract *Grahasphuṭānayane vikṣepavāsanā*. It is herein that Nīlakaṇṭha identifies the *śīghrocca* with the physical mean Sun and also gives the relations (F.29a) and (F.29b) between the ratio of the radii of the *śīghra* epicycle and the concentric with the ratio of the Earth–planet and Earth–Sun distances. Since the size of *śīghra* epicycles have already been fixed (see the tabulated values of radii of *śīghra* epicycles both in traditional planetary theory and in Nīlakaṇṭha's model in Table F.3), there is no longer any freedom to introduce a separate new hypothesis for the determination of the *śīghrocca*–planet distances.

Therefore, Nīlakaṇṭha's relations (F.32) and (F.33) for the planetary distances (however revolutionary they may be in relation to the traditional planetary models) are not consistent with the cosmological model definitively stated by Nīlakaṇṭha in *Grahasphuṭānayane vikṣepavāsanā*. In fact, once the *śīghrocca* of all the planets is identified with the physical mean Sun, the planetary distances get completely determined by the dimensions of the *śīghra* epicycles which are related to the ratios of the mean Sun–planet and Earth–Sun distances. The true Earth-planet distances in *yojanas* would then be given by the following:

$$\textit{Sphuṭa-kakṣyā} = \frac{\textit{kakṣyāvyāsārdha} \text{ of the Sun} \times \textit{śīghra-karṇa}}{\text{Radius of } \textit{śīghra} \text{ epicycle}} \quad \text{[ext.]} \tag{F.34}$$

$$\textit{Sphuṭa-kakṣyā} = \frac{\textit{kakṣyāvyāsārdha} \text{ of the Sun} \times \textit{śīghra-karṇa}}{\text{Radius}} \quad \text{[int.].} \tag{F.35}$$

The above relations follow from the fact that the mean orbit of the Sun is the *śīghra* epicycle in the case of the exterior planet, while it would be the concentric in the case of the interior planet.

It would be interesting to see whether any of the later works of Nīlakaṇṭha (which are yet to be located) or any of the works of later Kerala astronomers deal with these implications of the cosmological model of Nīlakaṇṭha for the calculation of planetary distances.

F.8 Annexure: Keplerian model of planetary motion

The planetary models described above can be appreciated better if we understand how the geocentric coordinates of a planet are calculated in Kepler's model. The three laws of planetary motion discovered by Kepler in the early seventeenth century, which form the basis of our present understanding of planetary orbits, may be expressed as follows:

1. Each planet moves around the Sun in an ellipse, with the Sun at one of the foci.
2. The areal velocity of a planet in its orbit is a constant.
3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of the ellipse in which it moves.

Kepler's laws can be derived from Newton's second law of motion and the law of gravitation. It may be recalled that Kepler's laws are essentially kinematical laws, which do not make any reference to the concepts of 'acceleration' and 'force', as we understand them today. Even then, they capture the very essence of the nature of planetary orbits and can be used to calculate the planetary positions, once we know the parameters of the ellipse and the initial coordinates. Since the planetary models proposed in Indian astronomy are also kinematical in nature, it makes sense to compare the two. So in what follows we will attempt to summarize the computation of the geocentric longitude and latitude of a planet which follows from Kepler's laws. This will also help in understanding the similarity that exists between the Keplerian model and the computational scheme adopted by the Indian astronomers.

F.8.1 Elliptic orbits and the equation of centre

A schematic sketch of the elliptic orbit of a planet P, moving around the Sun S with the latter at one of its foci is shown in Fig. F.10. Here a and b represent the semi-major and semi-minor axes of the ellipse. Γ refers to the first point of Aries. $\theta_a = \Gamma\hat{S}A$ denotes the longitude of the aphelion (A) and $\theta_h = \Gamma\hat{S}P$ is the heliocentric longitude of the planet.

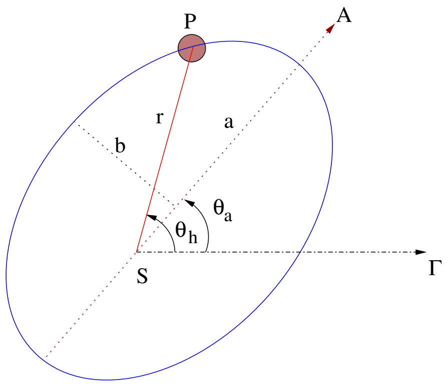

Fig. F.10 Elliptic orbit of a planet around the Sun.

The equation of the ellipse (in polar coordinates, with the origin at one of the foci), may be written as

$$\frac{l}{r} = 1 - e\cos(\theta_h - \theta_a), \qquad \text{(F.36)}$$

where e is the eccentricity of the ellipse and $l = a(1-e^2)$. Therefore

$$r = l[1 + e\cos(\theta_h - \theta_a)] + O(e^2),$$

$$r^2 = l^2[1 + 2e\cos(\theta_h - \theta_a)] + O(e^2). \tag{F.37}$$

As the area of an ellipse is πab, the areal velocity can also be written as $\frac{\pi ab}{T} = \frac{\omega ab}{2}$, where T is the time period and $\omega = \frac{2\pi}{T}$ is the mean angular velocity of the planet. Since the areal velocity of the planet at any instant is given by $\frac{1}{2}r^2\dot{\theta}_h$, and is a constant according to Kepler's second law, we have

$$r^2\dot{\theta}_h = \omega ab. \tag{F.38}$$

Using the above expression for r^2 in (F.37), we find

$$l^2\dot{\theta}_h[1 + 2e\cos(\theta_h - \theta_a)] = \omega ab + O(e^2). \tag{F.39}$$

Now $l = a\,(1 - e^2) = a + O(e^2)$ and $ab = a^2 + O(e^2)$. Hence

$$\dot{\theta}_h[1 + 2e\cos(\theta_h - \theta_a)] \approx \omega, \tag{F.40}$$

where the equation is correct to $O(e)$. Integrating with respect to time, we obtain

$$\begin{aligned} &\theta_h + 2e\sin(\theta_h - \theta_a)] \approx \omega t, \\ \text{or} \qquad &\theta_h - \omega t = -2e\sin(\theta_h - \theta_a). \end{aligned} \tag{F.41}$$

The argument of the sine function in the above equation involves θ_h, the actual heliocentric longitude of the planet, which is to be determined from the mean longitude θ_0. However, θ_h may be expressed in terms of θ_0 to $O(e^2)$. On so doing, the above equation reduces to

$$\theta_h - \omega t = \theta_h - \theta_0 = -2e\sin(\theta_0 - \theta_a) + O(e^2). \tag{F.42}$$

It may be noted that in (F.42) we have written ωt as θ_0, as the mean longitude of the planet increases linearly with time, t. $\theta_0 - \theta_a$, the difference between the longitudes of the mean planet and the apogee/aphelion, is known as the 'anomaly'. It may be noted that this difference is termed the *manda-kendra* in Indian astronomy. Thus (F.42) gives the equation of centre which is the difference between the true heliocentric longitude θ_h and the mean longitude θ_0, correct to $O(e)$, in terms of the anomaly. It is straightforward to see that the equation of centre correction arises owing to the eccentricity of the orbit and that its magnitude depends upon the value of the anomaly.

F.8.2 Geocentric longitude of an exterior planet

The orbits of all the planets are inclined at small angles to the plane of the Earth's orbit around the Sun, known as the ecliptic. We will ignore these inclinations and assume that all the planetary orbits lie on the plane of the ecliptic while calculating the planetary longitudes, as the corrections introduced by these inclinations are known to be small. We will consider the longitude of an exterior planet, i.e. Mars, Jupiter or Saturn, first and then proceed to discuss separately the same for an interior planet, i.e. Mercury or Venus.

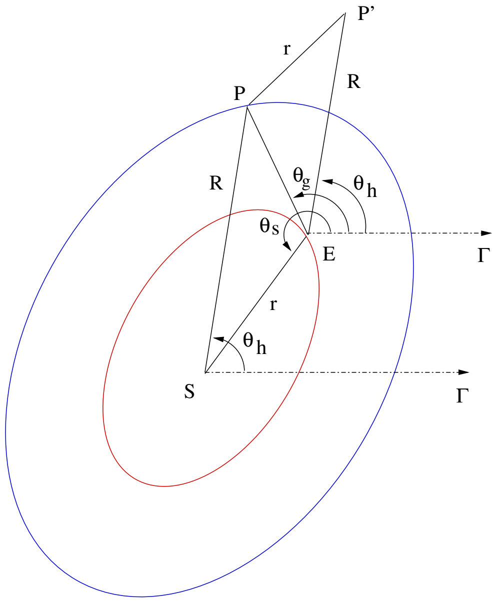

Fig. F.11 Heliocentric and geocentric longitudes of an exterior planet in Kepler's model.

The elliptic orbit of an exterior planet P and that of the Earth E around the Sun S are shown in Fig. F.11. Here, $\theta_h = \Gamma\hat{S}P$ is the true heliocentric longitude of the planet. $\theta_S = \Gamma\hat{E}S$ and $\theta_g = \Gamma\hat{E}P$ are the true geocentric longitudes of the Sun and the planet respectively, while r and R are the distances of the Earth and the planet from the Sun, which vary along their orbits.

We draw $EP' = R$ parallel to SP. Then, by construction, $P'P = r$ is parallel to ES. In the previous section (see (F.42)) it was described how θ_h is computed from the mean longitude θ_0, by applying the equation of centre. Now we need to obtain the true geocentric longitude θ_g from the heliolcentric longitude θ_h. It may be noted that

$$E\hat{P}S = P\hat{E}P' = \theta_g - \theta_h \qquad \text{and} \qquad E\hat{S}P = 180° - (\theta_s - \theta_h). \tag{F.43}$$

In the triangle ESP,

$$EP^2 = R^2 + r^2 - 2rR\cos[180° - (\theta_s - \theta_h)],$$
$$\text{or} \qquad EP = [(R + r\cos(\theta_s - \theta_h))^2 + r^2\sin^2(\theta_s - \theta_h)]^{\frac{1}{2}}. \tag{F.44}$$

Also,

$$\frac{\sin(E\hat{P}S)}{ES} = \frac{\sin(E\hat{S}P)}{EP}. \tag{F.45}$$

Using (F.43)–(F.44), we have

$$\sin(\theta_g - \theta_h) = \frac{r\sin(\theta_s - \theta_h)}{[(R + r\cos(\theta_s - \theta_h))^2 + r^2\sin^2(\theta_s - \theta_h)]^{\frac{1}{2}}}. \tag{F.46}$$

Here $(\theta_s - \theta_h)$, the difference between the longitude of the Sun and that of the heliocentric planet, is known as the 'solar anomaly' or 'anomaly of conjunction'.[50] Thus (F.46) gives $(\theta_g - \theta_h)$ in terms of the solar anomaly. Adding this to θ_h, we get the true geocentric longitude θ_g of the planet.

F.8.3 Geocentric longitude of an interior planet

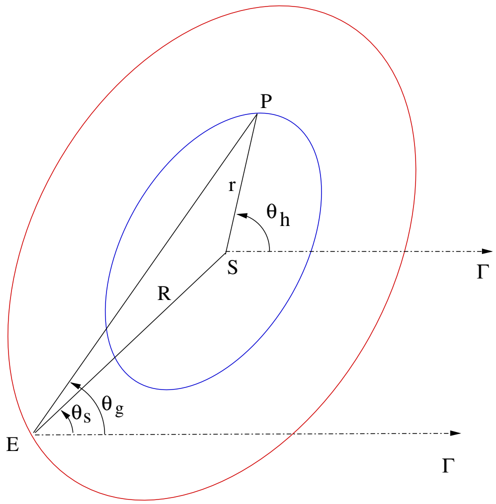

Fig. F.12 Heliocentric and geocentric longitudes of an interior planet in Kepler's model.

The elliptic orbit of an interior planet P and that of the Earth E around the Sun are shown in Fig. F.12. Here, $\theta_h = \Gamma\hat{S}P$ is the true heliocentric longitude of the planet, which can be computed from the mean heliocentric longitude and the equation of centre (see (F.42)). $\theta_s = \Gamma\hat{E}S$ and $\theta_g = \Gamma\hat{E}P$ are the true geocentric longitudes of the Sun and the planet respectively. As in the case of exterior planets, here too r

[50] The equivalent of this in Indian astronomy is the difference between the longitude of the *manda-sphuṭa* θ_{ms} and that of the *śīghrocca* θ_s, known as the *śīghra-kendra*.

and R represent the variable distances of the planet and the Earth from the Sun respectively.

It can easily be seen that

$$S\hat{E}P = \theta_g - \theta_s \qquad \text{and} \qquad E\hat{S}P = 180^\circ - (\theta_h - \theta_s). \tag{F.47}$$

Now considering the triangle ESP, we have

$$EP = [(R + r\cos(\theta_h - \theta_s))^2 + r^2 \sin^2(\theta_h - \theta_s)]^{\frac{1}{2}}. \tag{F.48}$$

Also,

$$\frac{\sin(S\hat{E}P)}{SP} = \frac{\sin(E\hat{S}P)}{EP}. \tag{F.49}$$

Using (F.47)–(F.49), we get

$$\sin(\theta_g - \theta_s) = \frac{r\sin(\theta_h - \theta_s)}{[(R + r\cos(\theta_h - \theta_s))^2 + r^2 \sin^2(\theta_h - \theta_s)]^{\frac{1}{2}}}. \tag{F.50}$$

Since all the parameters in the RHS of the above equation are known, the difference $(\theta_g - \theta_s)$ can be determined from this equation. Adding θ_s to this, we get the true geocentric longitude, θ_g of the planet. We now proceed to explain how the latitude of a planet is obtained in the Keplerian model.

F.8.4 Heliocentric and geocentric latitudes of a planet

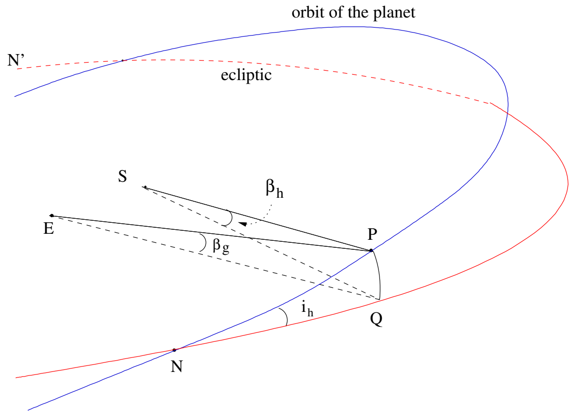

Fig. F.13 Heliocentric and geocentric latitudes of a planet in Kepler's model.

In Fig. F.13, the orbit of the planet P is shown to be inclined at an angle i_h to the ecliptic. N and N' are the nodes of the planetary orbit. PQ is the circular arc perpendicular to the ecliptic. Then the heliocentric latitude β_h is given by

$$\beta_h = \frac{PQ}{SP}. \tag{F.51}$$

If λ_P and λ_N are the heliocentric longitudes of the planet and the node, it can easily be seen that

$$\sin\beta_h = \sin i_h\ \sin(\lambda_P - \lambda_N) \qquad \text{or} \qquad \beta_h \approx i_h\ \sin(\lambda_P - \lambda_N), \tag{F.52}$$

as i_h and β_s are small. In the figure we have also shown the location of the Earth E. The latitude β_g (geocentric latitude) as measured from E would be different from the one measured from the Sun and is given by

$$\beta_g = \frac{PQ}{EP}. \tag{F.53}$$

From (F.51)–(F.53), we find that

$$\begin{aligned}\beta_g &= \beta_h \frac{SP}{EP} \\ &= \frac{i_h\ SP\ \sin(\lambda_P - \lambda_N)}{EP},\end{aligned} \tag{F.54}$$

where EP, the true distance of the planet from the Earth, can be found from (F.44) or (F.48).